

Geometry and Higher Category Theory

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Contents

Geometry and Higher Category Theory	5
Preface	5
1. Lecture I: Introduction	6
2. Lecture II: A Category of Categories that contains itself	16
3. Lecture III: Algebra with Categories	21
4. Lecture IV: Quasicoherent Sheaves of Higher Categories, I	29
Appendix to Lecture IV: Dualizable Categories	36
5. Lecture V: Quasicoherent Sheaves of Higher Categories, II	43
6. Lecture VI: Stefanich Rings	47
7. Lecture VII: Gestalten	60
8. Lecture VIII: Examples	66
Appendix to Lecture VIII: Descendability	71
9. Lecture IX: Six Functors and Gestalten	76
Appendix to Lecture IX: Six Functors on Gestalten	86
10. Lecture X: Cartier Duality	89
11. Lecture XI: Proof of Cartier Duality, I	99
12. Lecture XII: Proof of Cartier Duality, II	105
13. Lecture XIII: The Gestalt of Berkovich motives	111
Bibliography	119

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Preface

These are lecture notes for a course in Winter 2025/26, on joint work with Germán Stefanich.

Acknowledgments. I have profited tremendously from discussions with many mathematicians over the years. The influence of Lurie’s works on everything we do here is obvious, and without his foundational works this would be literally unthinkable. Working with Clausen on analytic stacks, this “extreme Tannakian” perspective had already emerged in a tentative way — although until recently I had remained too afraid of categorifying all the way — and he taught me a great deal about how to think about these things. More recently, the work of my current PhD students Aoki and Dauser has an enormous influence — their work has pushed me to seriously think about higher categories, and in some sense I simply tried to understand what is happening in their work. Moreover, our results on Cartier duality are heavily inspired by discussions with Rodríguez Camargo. I also wish to thank Johannes Anschütz, David Ben-Zvi, Sasha Efimov, Stavros Garoufalidis, Dennis Gaitsgory, David Hansen, Arthur-César le Bras, Lucas Mann and Sam Raskin.

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1. Lecture I: Introduction

A central theme throughout much of mathematics, and certainly in algebraic geometry, is the duality between geometry and algebra. Let us recall several examples that motivated the development of the subject.

EXAMPLE 1.1 (Gelfand). There is a duality (i.e., contravariant equivalence)

$$\begin{array}{ccc}
 \text{compact Hausdorff spaces} & \longleftrightarrow & \text{commutative } C^*\text{-algebras} \\
 X & \longmapsto & \text{Cont}(X, \mathbb{C}) \\
 \text{Hom}(A, \mathbb{C}) & \longleftarrow & A
 \end{array}$$

Here CompHaus is the category of compact Hausdorff topological spaces. A key ingredient in the equivalence is Urysohn's lemma that ensures the existence of a sufficient supply of continuous real- or complex-valued functions on compact Hausdorff spaces.

EXAMPLE 1.2 (Stone). There is a duality

$$\begin{array}{ccc}
 \text{profinite sets} & \longleftrightarrow & \text{boolean algebras} \\
 X & \longmapsto & \text{Cont}(X, \mathbb{F}_2) \\
 \text{Hom}(A, \mathbb{F}_2) & \longleftarrow & A
 \end{array}$$

Here, the category of profinite sets is equivalently that of totally disconnected compact Hausdorff spaces; and boolean algebras are characterized by the equation $x^2 = x$ for all $x \in A$.

EXAMPLE 1.3 (Hilbert). Let k be an algebraically closed field. There is a duality

$$\begin{array}{ccc}
 \text{affine } k\text{-varieties} & \longleftrightarrow & \text{integral finitely generated } k\text{-algebras} \\
 X & \longmapsto & \{\text{regular functions } X \rightarrow k\} \\
 \text{Hom}(A, k) & \longleftarrow & A
 \end{array}$$

Here k -varieties are what is now known as (separated) integral k -schemes of finite type. The essential content here is Hilbert's Nullstellensatz, ensuring that finitely generated k -algebras A admit sufficiently many maps to k .

EXAMPLE 1.4 (Grothendieck). There is a duality

$$\begin{array}{ccc}
 \text{affine schemes} & \longleftrightarrow & \text{commutative rings} \\
 X & \longmapsto & \{\text{regular functions on } X\} \\
 \text{Spec}(A) & \longleftarrow & A
 \end{array}$$

Unlike the previous examples, this duality is in its essence not a theorem, but a definition: During the development of algebraic geometry, the class of commutative rings that were allowed as duals of geometric objects became more and more general. At some point, this trend was taken to its logical conclusion, to allow all commutative rings; and to simply declare a notion of “affine scheme” as being the corresponding geometric notion. Several things had to be sacrificed: For example, in the case of nonreduced schemes, “regular functions on X ” are not determined by their values at points. Also, $\text{Spec}(A)$ is not a set of maps from A to a fixed field anymore; rather, it consists of (equivalence classes of) maps from A to all possible fields.

We note that in the third and fourth example, this is not a perfect duality between geometry and algebra. Rather, one has to restrict to “affine” objects on the geometric side, and then general geometric objects are obtained through gluing affine objects. For example, global functions on a projective line $\mathbb{P}_k^1$ are just k , so one cannot distinguish $\mathbb{P}_k^1$ from a point using global functions.

This situation, of having certain basic “affine” building blocks, out of which one builds general geometric objects, is a recurring theme in many different contexts. A general framework for it is the theory of sites and topoi. Namely, given some category C — the “affine” objects — one puts a Grothendieck topology τ on C and then passes to the category of sheaves $\mathrm{Shv}_\tau(C)$ on C . This way, one can take arbitrary colimits of objects of C , i.e. one can “glue” objects of C together to build new geometric objects. The resulting category is much larger than that of schemes; but the resulting category has the advantage of being a topos. This leads to the following idea:

IDEA 1.5. *Any good category of geometric objects (in which one can take all colimits) should be a topos.*

However, it depends on the choice of τ , and there are many choices one can make in the case of affine schemes: Zariski, Nisnevich, étale, fppf, fpqc, but (in principle) also non-subcanonical ones like the h-, v-, or arc-topology. All of these and more, and their relations, find plenty of use in algebraic geometry, giving rise to a panoply of slightly different categories of geometric objects, all built from affine schemes.

In fact, through the consideration of moduli spaces — such as the moduli space of elliptic curves — it became natural to consider not only sheaves of sets, but more generally sheaves of groupoids, also known as stacks. Indeed, elliptic curves admit automorphisms, and if one forgets those automorphisms by passing to isomorphism classes, one loses the good sheaf properties. In fact, the trend of allowing ever more general types of gluings has continued and let one consider also sheaves of ∞ -groupoids (=anima). Lurie [Lur09b] has developed the corresponding generalization of sheaf and topos theory, leading to the notion of an ∞ -topos — very roughly, the $(\infty, 1)$ -category of sheaves of anima on some site. Slightly updating the previous idea, we thus get the following idea:

IDEA 1.6. *Any good (∞) -category of geometric objects (in which one take all colimits) should be an ∞ -topos.*

Coming back to our discussion of the duality between geometry and algebra, let us consider the simplest kinds of stacks, say over an algebraically closed field k . These are the ones with only one point, but some automorphism group G of that point, i.e. classifying spaces $X = */G$ of group schemes G . Certainly, it is impossible to have interesting functions on $X = */G$. But Tannaka observed that the category of quasicoherent sheaves on X — which is the same thing as the category of k -linear representations of G — still allows one to recover $X = */G$, at least if one remembers its symmetric monoidal tensor product. We recall here that the functor taking any commutative ring A to its symmetric monoidal presentable abelian category of A -modules $\mathrm{Mod}(A)$ (usually) satisfies descent for the topology τ , and hence extends to a functor $X \mapsto \mathrm{QCoh}(X)$ on all schemes, or even (higher) stacks X . One important example of a quasicoherent sheaf is the structure sheaf $\mathcal{O}_X \in \mathrm{QCoh}(X)$, being the unit object; its endomorphisms are exactly the ring of global functions $\Gamma(X, \mathcal{O}_X)$.

EXAMPLE 1.7 (Tannaka). Let k be an algebraically closed field, and assume that τ is the fpqc topology. There is a perfect duality

$$\begin{array}{ccc}
 \text{classifying stacks of affine group schemes} & \longleftrightarrow & \text{Tannakian categories} \\
 X = */G & \longmapsto & \text{QCoh}(X) = \text{Rep}(G) \\
 A \mapsto \text{Fun}_{\otimes}(C, \text{Mod}(A)) & \longleftarrow & (C, \otimes)
 \end{array}$$

Here, Tannakian categories are certain compactly generated symmetric monoidal abelian k -linear categories with $\text{End}(1) = k$ and such that the compact and the dualizable objects agree.¹ The inverse functor takes a Tannakian category $(C, \otimes)$ to the stack, i.e. sheaf of groupoids, taking any commutative ring A to the groupoid of colimit-preserving $\otimes$ -functors from C to A -modules.

This Tannakian duality principle turns out to work in a much broader setting. First, it captures virtually all schemes (namely, quasicompact quasiseparated ones):

THEOREM 1.8 ([BC14]). *The functor*

$$\begin{array}{ccc}
 \{\text{qcqs schemes}\} & \longrightarrow & \{\text{symmetric monoidal presentable abelian categories}\} \\
 X & \longmapsto & \text{QCoh}(X)
 \end{array}$$

is fully faithful.

Thus, at the expense of passing from commutative rings to symmetric monoidal presentable abelian categories, we can fully capture the geometry of (qcqs) schemes. In fact, this extends to a broad class of “1-affine” stacks (cf. also [Gai15], [BZFN10], [Lur05]):

THEOREM 1.9 ([Ste23]). *The functor*

$$\begin{array}{ccc}
 \{\text{qcqs Artin stacks with affine diagonal}\} & \longrightarrow & \{\text{symm. mon. pres. abelian categories}\} \\
 X & \longmapsto & \text{QCoh}(X)
 \end{array}$$

is fully faithful.

At this point, we have managed to capture most geometric objects by algebra again. However, this time the algebra is much more general: There are many symmetric monoidal presentable abelian categories $(C, \otimes)$. Examples include:

- (1) Almost rings in the sense of Faltings [Fal88], [GR03]. If R is commutative ring with a flat ideal $I \subset R$ such that $I^2 = I$, then one defines the class of “almost zero R -modules” as those that are killed by I . These form a $\otimes$ -ideal inside R -modules, and one gets the resulting quotient category of R -modules by R/I -modules — this is known as (I) -almost R -modules, and forms a symmetric monoidal presentable abelian category. In most cases, the unit of this category is not compact. This is in contrast to the categories arising from qcqs stacks above, which always have compact unit.
- (2) Certain categories of functional-analytic nature. Let us highlight two classes of examples: One is the theory of Ind-Banach spaces used in the work of Ben-Bassat–Kelly–Kremnizer [BBKK24]; the other is the theory of solid or liquid modules used in my work with Clausen [CS24]. These usually have compact unit and are compactly generated, but not

¹We do not endow our Tannakian categories with a fibre functor; dually, the classifying stacks are not pointed.

all compact objects are dualizable. In fact, some infinite-dimensional vector spaces are compact objects of these categories.

The class of examples from (1) has turned out to be extremely useful in p -adic Hodge theory, but has since found many uses also in other contexts — as one example that has been inspirational to me, let me highlight the work of Vaintrob [Vai17], connecting it to symplectic geometry.

The class of examples from (2) has been used by both groups of authors to do analytic geometry, essentially by following an idea of Deligne:

IDEA 1.10 (Deligne, [Del90]). *For any symmetric monoidal presentable abelian $\otimes$ -category C , one can try to develop “ C -algebraic geometry”² following the template above:*

- (1) *The category $\mathrm{CAlg}(C)$ of commutative algebras in $(C, \otimes)$ generalizes the category of commutative rings.*
- (2) *Formally declare “ C -affine schemes” as being $\mathrm{CAlg}(C)^{\mathrm{op}}$.*
- (3) *Define some Grothendieck topology on C -affine schemes; for example, one always has an analogue of the fpqc topology.*
- (4) *Pass to the category of sheaves.*

This idea has been developed in some situations. Applied to the almost ring setup above, it gives Faltings “almost mathematics”; applied to the functional-analytic setups, it yields theories of analytic geometry, both in complex-analytic settings and in rigid-analytic settings. However, it seems somewhat intractable in the most general case.

In fact, in the applications to analytic geometry, one finds that it is necessary to pass from abelian categories to their derived categories, regarded as stable ∞ -categories, and take a Grothendieck topology consisting of certain non-flat covers. Namely, to get a good theory of analytic spaces one has to allow non-flat gluings — the analogue of localization to open subsets turns out not to be flat. This precludes a good abelian category of quasicoherent sheaves, but it turns out that when one passes from abelian categories to derived categories, treated as stable ∞ -categories, the formalism works very well; in particular, one gets good “derived ∞ -categories of quasicoherent sheaves” in complex-analytic geometry and rigid-analytic geometry.

Thus, we are forced to generalize further to symmetric monoidal presentable stable ∞ -categories, or in more compact notation

$$C \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L).$$

Here Pr^L denotes Lurie’s ∞ -category of presentable ∞ -categories and colimit-preserving functors (and the subscript st indicates the subcategory of stable ∞ -categories). This has a symmetric monoidal tensor product $(C, D) \mapsto C \otimes D$, which classifies functors out of $C \times D$ that commute with colimits in each variable (i.e., “bilinear functors”). Passing to stable ∞ -categories, the unit object is the ∞ -category of “spectra” in the sense of algebraic topology; to avoid clash with the notion of spectrum of a ring, we will call these $\mathbb{S}$ -modules and denote their category by $D(\mathbb{S})$, where $\mathbb{S}$ denotes the unit object, usually known as the sphere spectrum. The subcategory of connective $\mathbb{S}$ -modules $D_{\geq 0}(\mathbb{S})$ is the ∞ -category of “commutative group objects in anima”, i.e. $\mathbb{E}_{\infty}$ -groups in anima. To pass to all of $D(\mathbb{S})$, one passes to the stabilization, i.e. one records an arbitrary object $M \in D(\mathbb{S})$ in terms of its truncations $\tau_{\geq -n}M$, which up to shift by n can be regarded as objects

²often called “relative” (to C) “algebraic geometry”

of $D_{\geq 0}(\mathbb{S})$. Concretely, objects of $D(\mathbb{S})$ are sequences $(M_0, M_1, \dots)$ of $M_n \in D_{\geq 0}(\mathbb{S})$ together with isomorphisms $M_n \cong \Omega(M_{n+1})$ of $\mathbb{E}_\infty$ -groups, where $\Omega(M) = * \times_M *$ is the loop space.³

It turns out that $\mathrm{Pr}_{\mathrm{st}}^L$ with its symmetric monoidal tensor product makes it possible to do “algebra with categories”. For example, the functor $A \mapsto D(A)$ satisfies

$$D(C) \otimes_{D(A)} D(B) \cong D(C \otimes_A B).$$

In particular, $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ is the ∞ -category of symmetric monoidal presentable stable ∞ -categories, and can be seen as the categorical version of commutative rings. In fact, via $A \mapsto D(A)$, commutative rings (over $\mathbb{S}$, i.e. $A \in \mathrm{CAlg}(D(\mathbb{S}))$) embed fully faithfully into $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$.

QUESTION 1.11. Is it reasonable to build a notion of geometry by starting with all of $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$, in place of commutative rings?

This question has been investigated to some extent. It is closely related to Balmer’s “tensor-triangular geometry” [Bal10]; see also [ABC⁺25] for a recent account.

PROBLEMS 1.12. *However, I want to highlight two problems with this proposal:*

- (1) *In the stated generality, the category $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ seems completely uncontrollable. In practice, one might restrict to compactly generated objects (often assuming that the compact and dualizable objects agree); this yields a subclass that seems controllable. However, the two examples mentioned — almost rings, and analytic rings — fall outside of this restricted class: In the almost setting, the category is not compactly generated, while in the analytic setting, the compact objects are not all dualizable. One may also want to combine them, for example in the form of “analytic almost rings”, cf. [Man22].*
- (2) *It does not restore a perfect duality between geometry and algebra — there will still be many geometric objects X that cannot be detected by their $D_{\mathrm{qc}}(X) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$, as in the example just below. Thus, it again becomes necessary to choose some Grothendieck topology, and pass to sheaves.*

EXAMPLE 1.13. Work with the étale topology on affine schemes over a field k , and consider the stack $B^2\mathbb{G}_m$, the 2-fold classifying stack of the multiplicative group $\mathbb{G}_m$. Then the pullback functor along $B^2\mathbb{G}_m \rightarrow *$ yields an equivalence

$$D(k) = D_{\mathrm{qc}}(*) \xrightarrow{\cong} D_{\mathrm{qc}}(B^2\mathbb{G}_m).$$

Note that $B^2\mathbb{G}_m$ is an object that arises in practice: Namely, for any scheme X , maps $X \rightarrow B^2\mathbb{G}_m$ correspond to $H_{\mathrm{ét}}^2(X, \mathbb{G}_m)$ which at least for regular X is precisely a Brauer class on X , so yields an Azumaya algebra $\mathcal{A} \in \mathrm{Alg}(\mathrm{QCoh}(X))$. The category of modules $\mathrm{Mod}_{\mathcal{A}}(D_{\mathrm{qc}}(X))$ is a sheaf of categories over X that is locally on X isomorphic to $D_{\mathrm{qc}}(X)$, but not always globally so.

We see that while $B^2\mathbb{G}_m$ does not admit enough quasicoherent sheaves, it does admit an interesting example of a sheaf of categories. Let us recall this notion. For any commutative ring A , we have $D(A) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$, and we can then pass to modules again, getting

$$\mathrm{Pr}_A^L := \mathrm{Mod}_{D(A)}(\mathrm{Pr}_{\mathrm{st}}^L),$$

³Often, this definition is given in a different form, where all M_n are just pointed anima, together with equivalences $M_n \cong \Omega(M_{n+1})$. The group structure on M_n then arises from its structure as the loop space of M_{n+1} ; as it is in fact an infinite loop space $M_n \cong \Omega(M_{n+1}) \cong \Omega^2(M_{n+2}) \cong \dots$, this is an $\mathbb{E}_\infty$ -group structure.

which is again a symmetric monoidal ∞ -category. We would like to say that it is a “symmetric monoidal presentable $(\infty, 2)$ -category”, but let us slightly postpone a discussion of the notion of presentable $(\infty, 2)$ -category.

The association $A \mapsto \mathrm{Pr}_A^L$ has strong descent properties, allowing one to extend it to a functor

$$X \mapsto \mathrm{Pr}_X^L$$

for any stack X , if τ is suitably chosen. The above example can now be refined by saying that there is a canonical $\otimes$ -invertible object in $\mathrm{Pr}_{B^2\mathbb{G}_m}^L$. This is the analogue of the canonical $\otimes$ -invertible object in

$$D_{\mathrm{qc}}(B\mathbb{G}_m) = D_{\mathrm{qc}}(*/\mathbb{G}_m),$$

namely the tautological line bundle on $*/\mathbb{G}_m$ (which is the classifying stack of line bundles). This in turn may be regarded as the analogue of the canonical invertible function on $\mathbb{G}_m$, which is just the coordinate function T .

We see that the geometry of $B^2\mathbb{G}_m$ can be detected in terms of the algebra of $\mathrm{Pr}_{B^2\mathbb{G}_m}^L$, passing up yet another categorical level, to “symmetric monoidal presentable $(\infty, 2)$ -categories”. However, if we pass to $B^3\mathbb{G}_m$, the issue comes back: Pullback along $B^3\mathbb{G}_m \rightarrow *$ yields an equivalence

$$\mathrm{Pr}_{B^3\mathbb{G}_m}^L \cong \mathrm{Pr}_k^L.$$

In this case, $B^3\mathbb{G}_m$ would support an “invertible sheaf of linear $(\infty, 2)$ -categories”, requiring one to categorify another level; and in general $B^n\mathbb{G}_m$ supports an “invertible sheaf of linear $(\infty, n-1)$ -categories”.

Thus, the only possibility to reflect all geometric objects in terms of (higher) algebra is to keep track of all higher categories of “sheaves of linear (∞, n) -categories”. These should be symmetric monoidal presentable $(\infty, n+1)$ -categories.

In Stefanich’s thesis [Ste21], such a theory of presentable (∞, n) -categories has been developed. However, recently Aoki [Aok25a] has observed that it is not as well-behaved as Stefanich conjectured — in particular, presentable (∞, n) -category do not admit limits in general. In a related vein, the expected forms of adjoint functor theorems do not hold true. The basic reason are the usual size issues of category theory: If one wants to form a “category of categories”, one runs into size issues, so one usually fixes some universes and works with a large category of small categories. This will however mean that the “category of all categories” does not contain itself as an object. If one iterates this procedure, one will likely have to fix different universes for the different categorical levels in place. The choice of all these different universes makes the resulting notions feel somewhat unnatural, and hard to work with.

But Aoki has also proposed a fix, which actually simplifies the theory. Namely, recall that any presentable ∞ -category C is, by definition, κ -compactly generated for some regular cardinal κ . This way, one can write Pr^L as a union of non-full subcategories

$$\mathrm{Pr}_\kappa^L \subset \mathrm{Pr}^L$$

consisting of the κ -compactly generated presentable ∞ -categories, and functors preserving κ -compact objects. The case $\kappa = \aleph_0$ corresponds to compactly generated categories, which accounts for many but not all categories used in mathematical practice; but already the next case $\kappa = \aleph_1$ corresponds to “countably compactly generated” categories, and this does include (virtually) all categories used in mathematical practice. In particular,

$$\mathrm{Pr}_{\aleph_1}^L \subset \mathrm{Pr}^L$$

is stable under countable limits, as well as all colimits and tensor products, so it is somewhat difficult to escape from this setup.

But if we decide to fix some κ — and we will usually just take $\kappa = \aleph_1$ — then Pr_κ^L is an ∞ -category that happens to, itself, be presentable and κ -compactly generated, i.e.:

FACT 1.14. $\mathrm{Pr}_\kappa^L \in \mathrm{Pr}_\kappa^L$.

(The sign $\in$ is a type-theoretic judgement here, not the literal $\in$ -relation of axiomatic ZFC.) To avoid clutter, let us write $1\mathrm{Pr} = \mathrm{Pr}_\kappa^L$ (where we stress that κ has been fixed from now on). As it is a symmetric monoidal category, we in fact have $1\mathrm{Pr} \in \mathrm{CAlg}(\mathrm{Pr}_\kappa^L) = \mathrm{CAlg}(1\mathrm{Pr})$. Thus, we can define:

DEFINITION 1.15 ([Aok25a]).

- (i) The symmetric monoidal $(\infty, 1)$ -category of κ -presentable $(\infty, 2)$ -categories is

$$2\mathrm{Pr} := \mathrm{Mod}_{1\mathrm{Pr}}(1\mathrm{Pr}).$$

This is, itself, a symmetric monoidal κ -presentable $(\infty, 1)$ -category, i.e. $2\mathrm{Pr} \in \mathrm{CAlg}(1\mathrm{Pr})$.

- (2) More generally, for any $n \geq 2$, the symmetric monoidal $(\infty, 1)$ -category of κ -presentable (∞, n) -categories is

$$n\mathrm{Pr} := \mathrm{Mod}_{(n-1)\mathrm{Pr}}(1\mathrm{Pr}) \in \mathrm{CAlg}(1\mathrm{Pr}).$$

REMARK 1.16. Although in informal terms, $n\mathrm{Pr}$ is an $(\infty, n+1)$ -category, we will usually ignore the non-invertible higher morphisms, and treat it merely as an $(\infty, 1)$ -category. The higher morphisms can be recovered as $n\mathrm{Pr}$ is tensored, and hence enriched, over itself. This has the nice consequence that on a technical level, we can mostly directly work with Lurie’s foundations on $(\infty, 1)$ -categories.

For any commutative ring A , we can now similarly inductively define the notion of “ A -linear presentable (∞, n) -categories”:

DEFINITION 1.17. For a commutative ring A , we set

$$1\mathrm{Pr}_A = \mathrm{Mod}_{D(A)}(1\mathrm{Pr}) \in \mathrm{CAlg}(2\mathrm{Pr})$$

and inductively

$$n\mathrm{Pr}_A = \mathrm{Mod}_{(n-1)\mathrm{Pr}_A}(n\mathrm{Pr}) \in \mathrm{CAlg}((n+1)\mathrm{Pr}).$$

More generally, for any $C \in \mathrm{CAlg}(1\mathrm{Pr})$, we set

$$1\mathrm{Pr}_C = \mathrm{Mod}_C(1\mathrm{Pr}) \in \mathrm{CAlg}(2\mathrm{Pr})$$

and inductively

$$n\mathrm{Pr}_C = \mathrm{Mod}_{(n-1)\mathrm{Pr}_C}(n\mathrm{Pr}) \in \mathrm{CAlg}((n+1)\mathrm{Pr}).$$

We will discuss the preceding definitions in more detail in the coming lectures. These are examples of the following general notion.

DEFINITION 1.18. Let StRing be⁴ the ∞ -category of sequences $(A_1, A_2, \dots)$ where

$$A_n \in \mathrm{CAlg}(n\mathrm{Pr})$$

⁴The name can be read simultaneously as an abbreviation for “Stefanich ring”, “stable ring”, and “string of categories”. I refer to them as “Stefanich rings”.

is a symmetric monoidal κ -presentable (∞, n) -category, together with equivalences

$$A_n \cong \mathrm{End}_{A_{n+1}}(1)$$

in $\mathrm{CAlg}(n\mathrm{Pr})$ for all $n \geq 1$.

Equivalently,

$$\mathrm{StRing} = \lim_{\mathrm{End}_{-}(1)} \mathrm{CAlg}(n\mathrm{Pr}).$$

REMARK 1.19. In the original lecture, we asked that A_1 is stable. Although this will generally be the setting of interest, much of the basic theory works without the assumption, so we decided to remove it. If A_1 is stable, one also gets the commutative ring

$$A_0 = \mathrm{End}_{A_1}(1) \in \mathrm{CAlg}(D(\mathbb{S}))$$

in $\mathbb{S}$ -modules.

REMARK 1.20. For $M = (M_0, M_1, \dots) \in D(\mathbb{S})$, the addition operation on M_0 is encoded in the deloopings $M_1, M_2, \dots$ via $M_0 \cong \Omega^n(M_n)$. Similarly, for $A = (A_0, A_1, \dots) \in \mathrm{StRing}$ (with A_1 stable, say), the multiplication operation on A_0 is encoded in the “deloopings” $A_1, A_2, \dots$. Namely, in the definition of StRing , we could equivalently just consider pointed $A_n \in n\mathrm{Pr}$ with equivalences $A_n \cong \mathrm{End}_{A_{n+1}}(*)$ (with the pointing of A_n corresponding to the identity). Then the symmetric monoidal structure of A_n arises from its structure as an iterated endomorphism category, and in particular the multiplication on A_0 comes from

$$A_0 = \mathrm{End}_{A_1}(1) = \mathrm{End}_{\mathrm{End}_{A_2}(1)}(1) = \dots$$

In this sense, StRing arises from the usual notion of a commutative ring by “taking full deloopings of both addition and multiplication”.

Consider the functor from commutative rings to StRing given by $A \mapsto (A, D(A), 1\mathrm{Pr}_A, 2\mathrm{Pr}_A, \dots)$. For a suitable Grothendieck topology τ , this extends to stacks, giving a functor from stacks to $\mathrm{StRing}^{\mathrm{op}}$. This functor is close to being conservative. In fact, in the context of analytic stacks [CS24] one can be much more precise:

FACT 1.21. *Consider the functor*

$$\mathrm{PShv}(\mathrm{AnRing}^{\mathrm{op}}) \rightarrow \mathrm{StRing}^{\mathrm{op}} : X \mapsto (\mathcal{O}(X), D_{\mathrm{qc}}(X), 1\mathrm{Pr}_X, 2\mathrm{Pr}_X, \dots)$$

obtained via left Kan extension. Then a map $f : X \rightarrow Y$ of $\aleph_1$ -compact objects in $\mathrm{PShv}(\mathrm{AnRing}^{\mathrm{op}})$ becomes an isomorphism in $\mathrm{AnStack}$ if and only if it becomes an isomorphism in $\mathrm{StRing}^{\mathrm{op}}$. In other words, the functor factors as

$$\mathrm{PShv}(\mathrm{AnRing}^{\mathrm{op}}) \rightarrow \mathrm{AnStack} \rightarrow \mathrm{StRing}^{\mathrm{op}}$$

*where the second functor is conservative on $\aleph_1$ -compact objects.*⁵

REMARK 1.22. The definition of $\mathrm{AnStack}$ in [CS24] is subtle, as it is an ∞ -topos between sheaves and hypersheaves on AnRing for some Grothendieck topology. The above fact gives the clearest explanation for our precise choice of definition of $\mathrm{AnStack}$.⁶

⁵The restriction to $\aleph_1$ -compact objects is harmless in practice — all stacks one ever considers are countable colimits of affines.

⁶In fact, the definition in the lectures [CS24] was slightly wrong — it did not define an ∞ -topos — but the corrected definition given in [ABL⁺25, Definition 4.2.1] is equivalent to the above.

We see that passing from geometry to algebra in this sense is conservative. Thus, the following question becomes sensible.

QUESTION 1.23. Is it reasonable to build a notion of geometry by starting with all of StRing in place of commutative rings?

In 1.12, we mentioned two problems with a similar proposal. Let us analyze them in this new context:

- (1) Obviously StRing is even more uncontrollable than its full subcategory $\text{CAlg}(\mathbf{1PrS})$; we certainly did not improve on this front.
- (2) It turns out that it is no longer necessary to choose a Grothendieck topology, or pass to sheaves!

Namely:

THEOREM 1.24 (S.–Stefanich). *For all practical purposes, the $(\infty, 1)$ -category $\text{StRing}^{\text{op}}$ is an ∞ -topos in the sense of Lurie.*

REMARK 1.25. We will make the qualifier “For all practical purposes” more precise later in the lectures. Part of it has to do with our choice of cutoff cardinal κ — for example, as StRing is a presentable ∞ -category, its opposite ∞ -category cannot also be presentable.

In particular, $\text{StRing}^{\text{op}}$ carries a canonical Grothendieck topology, namely the one where covers are the effective epimorphisms of this topos. And as it already is a topos, we do not need to adjoin any new colimits. Thus, we restore a perfect duality between geometry and algebra!

Obviously, one needs a name for the corresponding geometric objects. They are some higher notion of “stacks”, but we believe they should not carry a derivative name, but rather are their own kind of thing. We settled for the following German word, meaning “shape, form, figure”.

DEFINITION 1.26. The $(\infty, 1)$ -category of *Gestalten* is

$$\text{Gest} := \text{StRing}^{\text{op}}.$$

For $A \in \text{StRing}$, we denote by $\text{Gest}(A) \in \text{Gest}$ the corresponding Gestalt.

The goal of the lectures is to show that this is a workable theory of geometric objects, and to give many examples.

EXAMPLE 1.27.

- (1) By the above, there is a functor from algebraic stacks or also analytic stacks towards Gestalten. By Tannakian theorems, these functors are fully faithful on certain reasonable large subcategories. However, they fail to be fully faithful in general. For example, if $\mathbb{R}_{>0}$ denotes the internal real numbers object of $\text{Gest}/_{\text{Gest}(D(\mathbb{Z}))}$ (which in fact comes from the Betti stack $\mathbb{R}_{>0, \text{Betti}}$ via the map from algebraic stacks to Gestalten), then there is a nontrivial map

$$* \rightarrow */_{\mathbb{R}_{>0}}$$

in Gestalten that does not come from a map of algebraic stacks. This comes from the discussion in [Sch25c].

- (2) One can define a 6-functor formalism on Gestalten, for a certain class E of !-able maps, and hence define many geometric notions inherent to any 6-functor formalism, such as open immersions, (cohomologically) étale maps, (cohomologically) smooth maps, (cohomologically) proper maps, etc. This pulls back to the 6-functor formalism on analytic stacks.
- (3) Recall from [Sch25b] that empirically, many 6-functor formalisms $D : C^{\text{op}} \rightarrow 1\text{Pr}$ factor over analytic stacks as

$$C^{\text{op}} \rightarrow \text{AnStack}^{\text{op}} \xrightarrow{D_{\text{qc}}} 1\text{Pr}.$$

Replacing analytic stacks by Gestalten, it turns out (cf. [Ste25a]) that for any presentable 6-functor formalism one can construct a factorization as

$$C^{\text{op}} \rightarrow \text{Gest}^{\text{op}} = \text{StRing} \xrightarrow{A \mapsto A_1} 1\text{Pr},$$

for some functor

$$C \rightarrow \text{Gest} : X \mapsto [X]_D.$$

In different language, any six-functor formalism on C yields a transmutation $C \rightarrow \text{Gest}$, $X \mapsto [X]_D$, commuting with finite limits, and with gluings with respect to the !-topology of [Sch25b, Appendix to Lecture V].

In particular, the motivic 6-functor formalism SH of Morel–Voevodsky [MV99] defines a Gestalt $[\text{Spec}(\mathbb{Z})]_{\text{SH}}$. The following theorem will appear in Aoki’s thesis [Aok25b], and makes precise the idea that motives are related to ring stacks (or rather Ringgestalten), cf. [Sch24]. We mention it here as an example of the new kinds of geometry enabled by the formalism we want to develop in these lectures; and as an example that despite the immense generality, one can still give clean statements.

THEOREM 1.28 (Aoki). *The Gestalt $[\text{Spec}(\mathbb{Z})]_{\text{SH}}$ represents the functor taking a Gestalt X over $\text{Gest}(\mathbb{S})$ to the groupoid of Ringgestalten R over X such that*

- (1) *The map $R \rightarrow X$ is 1-affine, cohomologically smooth, and homologically contractible, and*
- (2) *the zero section $X \xrightarrow{0} R$ is a closed immersion, the units $R^\times \hookrightarrow R$ form an open immersion, and together they form a stratification of R .*

2. Lecture II: A Category of Categories that contains itself

In this lecture, we start to make some of the definitions from the introductory lecture precise. As is clear from that lecture, we will have to talk about “categories of categories of ... of categories”. The standard approach would be to fix a universe level each time – i.e., there is a “large category of small categories”, a “very large category of large categories of small categories”, etc. I feel very uncomfortable with all those choices of universe levels, and I believe it also becomes unwieldy to work in such a formalism. Instead, in this lecture, we will present a formalism where this universe-juggling is not required, and instead we have a “category of categories” that literally contains itself.

As this lecture deals with certain foundational issues, I should make some clarifications about my metatheory. I want to speak about classes, so let us work in the von Neumann–Bernays–Gödel axioms NBG; recall that this is basically a repackaging of ZFC which allows proper classes besides sets. Below, I will consider various categories or $(\infty, 1)$ -categories. I will allow the objects of an $(\infty, 1)$ -category to form a proper class, but all morphisms sets (or anima) are asked to be (essentially) small, i.e. the $(\infty, 1)$ -category is locally small. This is the case, in particular, for $\mathbf{Set}$, or also anima $\mathbf{Ani}$.

Let me now start in earnest. It has been observed already by Cantor that the “collection of all sets” cannot itself be a set – it is too large for this to be the case. But structurally, there is another reason: the “collection of all sets” should not even be a set, rather it should be a category. Namely, structurally, it does not make sense to ask for two sets to be equal; one can only consider the groupoid of all sets as a natural mathematical object. So there are two issues with trying to consider the “collection of all sets” as a set itself: One of size, and one of structural nature – it should rather be a category (or at least a groupoid, neglecting non-invertible maps).

Focusing on the structural aspect for the moment, the “collection of all groupoids” or “collection of all categories” should itself not be a groupoid or category – rather, it should be a 2-groupoid or 2-category. This process continues, leading, at least intuitively, to notions of n -categories for all n , up to and including $n = \infty$. It is useful to have the following slightly finer notion:

DEFINITION 2.1. For $0 \leq m \leq n \leq \infty$, an (n, m) -category is an n -category in which all k -morphisms are invertible for $k > m$.

For example, an $(n, 0)$ -category is the same as an n -groupoid, which (by the homotopy hypothesis, or by definition) is the same thing as an n -truncated anima. Building on ideas of Boardman–Vogt and Joyal, Lurie has developed an extensive theory of $(\infty, 1)$ -categories [Lur09b].

Now passing to the limit $n = \infty$, structurally the “collection of all ∞ -groupoids” has the right to itself be an ∞ -groupoid. It has slightly more structure, namely that of an $(\infty, 1)$ -category, but we are free to forget the non-invertible maps. Similarly, the “collection of all $(\infty, 1)$ -categories” is most naturally an $(\infty, 2)$ -category, but we are free to forget the non-invertible 2-morphisms and treat it as an $(\infty, 1)$ -category.

However, the size issues remain – in our chosen setup where objects may form a proper class, there is an $(\infty, 1)$ -category $\mathbf{Cat}_\infty$ of small $(\infty, 1)$ -categories, but it is not itself small.

On the other hand, the examples of $(\infty, 1)$ -categories we would like to organize into an $(\infty, 1)$ -category are themselves large – for example, for any ring A , we would like to consider the $(\infty, 1)$ -category $D(A)$. Thus $\mathbf{Cat}_\infty$ does not only not contain itself as an object, it contains none of the relevant examples as objects. However, while the examples like $D(A)$ are large, they are controlled by a small subcategory: There is the full subcategory $\mathbf{Perf}(A)$ of perfect complexes (i.e., those that

can be represented by finite complexes of finite projective A -modules), which is essentially small, and $D(A) = \text{Ind}(\text{Perf}(A))$ is gotten by formally adding all filtered colimits. This is a general pattern for virtually all large categories arising in practice, and axiomatized in the following definition of “presentability”.

DEFINITION 2.2. Let C be an $(\infty, 1)$ -category.⁷ Assume that C admits all small colimits.

- (1) An object $X \in C$ is compact if $\text{Hom}_C(X, -) : C \rightarrow \mathbf{Ani}$ commutes with filtered colimits.
- (2) More generally, if κ is a regular cardinal, an object $X \in C$ is κ -compact if $\text{Hom}_C(X, -) : C \rightarrow \mathbf{Ani}$ commutes with κ -filtered colimits.
- (3) The $(\infty, 1)$ -category C is κ -presentable if there is a set S of κ -compact objects of C such that every object of C can be written as a small colimit of a diagram with values in S .
- (4) The $(\infty, 1)$ -category C is presentable if it is κ -presentable for some regular cardinal κ .

REMARK 2.3. We recall some notions used here.

- (1) A cardinal κ is regular if the union of $< \kappa$ many sets of size $< \kappa$ is still less than κ . This is the case for $\kappa = \aleph_0$, and if $\kappa = \lambda^+$ is the successor cardinal of some λ , which will be the cases of interest to us.
- (2) A small $(\infty, 1)$ -category C is κ -filtered if for any $(\infty, 1)$ -category D of size $< \kappa$, any map $D \rightarrow C$ can be extended to $D^\triangleright \rightarrow C$ (where $D^\triangleright$ is the $(\infty, 1)$ -category obtained from D by freely adjoining a final object); in other words, from the perspective of κ -small categories, C looks like it has a final object.
- (3) The notion of κ -compact objects should be regarded as “ $< \kappa$ -presented”. We will mostly care about $\kappa = \aleph_1$ (often written as the ordinal ω_1), in which case this is the notion of countable presentation.

THEOREM 2.4. Let C be a κ -presentable $(\infty, 1)$ -category. Then the subcategory $C^\kappa \subset C$ of κ -compact objects is essentially small, and the natural functor

$$\text{Ind}_\kappa(C^\kappa) \rightarrow C$$

is an equivalence, where Ind_κ freely adjoins κ -filtered colimits.

PROOF. First, we check fully faithfulness. Consider formal colimits “ $\text{colim}_i X_i$ ” and “ $\text{colim}_j Y_j$ ” in $\text{Ind}_\kappa(C^\kappa)$. Then

$$\text{Hom}_C(\text{colim}_i X_i, \text{colim}_j Y_j) = \lim_i \text{Hom}_C(X_i, \text{colim}_j Y_j) = \lim_i \text{colim}_j \text{Hom}_C(X_i, Y_j)$$

where the first equality is by definition of colimits, and the second follows from κ -compactness of all X_i . But the last term is exactly what morphisms in $\text{Ind}_\kappa(C^\kappa)$ are defined to be.

For essential surjectivity, let $D \in C$ be any object. Write D as a colimit $\text{colim}_{i \in I} X_i$ of objects $X_i \in S$, where S is a set of κ -compact objects as in the definition. Think of I as a simplicial set as in the official definition of $(\infty, 1)$ -categories as quasicategories. Write I as the union of its κ -small subsimplicial sets $I' \subset I$. As κ is regular, the collection of such I' is a κ -filtered partially ordered set. We can then write

$$\text{colim}_{i \in I} X_i = \text{colim}_{I' \subset I} (\text{colim}_{i \in I'} X_i).$$

Here, the inner colimit $\text{colim}_{i \in I'} X_i$ is a κ -small colimit of κ -compact objects and thus itself κ -compact; while the outer colimit is κ -filtered. This shows that $\text{Ind}_\kappa(C^\kappa) \rightarrow C$ is essentially surjective, and hence the equivalence.

⁷As stated above, we always require that C is locally small.

It remains to see that C^κ is essentially small. But if D above is κ -compact, then the isomorphism

$$D = \operatorname{colim}_{I' \subset I} (\operatorname{colim}_{i \in I'} X_i)$$

factors over $\operatorname{colim}_{i \in I'} X_i$ for some κ -small $I' \subset I$, making D a retract of $\operatorname{colim}_{i \in I'} X_i$. But there is a set of worth of κ -small I' up to isomorphism, a set worth of functors $I' \rightarrow S$ for each I' (as C is locally small), and a set of worth of possible retracts of $\operatorname{colim}_{i \in I'} X_i$ for each $I' \rightarrow S$, showing that C^κ is essentially small. $\square$

REMARK 2.5. The terminology used to be “locally presentable category” (cf. e.g. [GU71]), emphasizing that each object of C can be presented in terms of the generating objects in C^κ . However, it makes sense to drop “locally”, as in fact C itself admits a presentation. Namely (cf. [Lur09b, Theorem 5.5.1.1 (5)]), C is presentable if and only if there is a small $(\infty, 1)$ -category S and a set R of maps in the $(\infty, 1)$ -category $\langle S \rangle$ freely generated by S under small colimits, such that C is the free $(\infty, 1)$ -category with all small colimits with a map from S and such that all maps in R become isomorphisms. Concretely, by [Lur09b, Theorem 5.1.5.6], $\langle S \rangle = \operatorname{Fun}(S^{\operatorname{op}}, \operatorname{Ani})$ via the Yoneda embedding, and inverting all maps in R corresponds to passing to the full subcategory of those functors $F : S^{\operatorname{op}} \rightarrow \operatorname{Ani}$ such that mapping any morphism in R to F yields an isomorphism.

In this perspective, C is κ -presentable if R consists of maps in the $(\infty, 1)$ -category generated by S under κ -small colimits.

REMARK 2.6. All of our categories of interest such as Set , Ring , Ani , $D(A)$ are presentable. In fact these examples are even κ -presentable for $\kappa = \aleph_0$; i.e., compactly generated. However, there are many interesting examples that are not compactly generated, for example almost modules, or categories of sheaves on manifolds. These are still “countably compactly generated”, i.e. κ -presentable for $\kappa = \aleph_1$. We will not run into κ -presentable categories for which $\kappa > \aleph_1$ is required (but of course it is easy to build examples if one is specifically looking for them).

Coming back to the discussion from the beginning, we would like to have a collection of all presentable $(\infty, 1)$ -categories, and ideally that should itself be a presentable $(\infty, 1)$ -category. Lurie makes the following definitions, passing up to a larger universe to avoid size issues; we will momentarily discuss these size issues.

DEFINITION 2.7. Let Pr^L be the $(\infty, 1)$ -category whose objects are presentable $(\infty, 1)$ -categories, and morphisms are colimit-preserving functors.⁸

For a regular cardinal κ , let $\operatorname{Pr}_\kappa^L \subset \operatorname{Pr}^L$ be the non-full subcategory whose objects are κ -presentable $(\infty, 1)$ -categories, and morphisms are those colimit-preserving functors that also preserve κ -compact objects.

REMARK 2.8. As each presentable $(\infty, 1)$ -category has a class of objects, we cannot collect those together as in NBG there are no “classes of classes”. A way out is to encode a presentable $(\infty, 1)$ -category C instead as a pair (κ, C^κ) consisting of a regular cardinal κ and a small $(\infty, 1)$ -category C^κ so that $C = \operatorname{Ind}_\kappa(C^\kappa)$; this does define a class. However, there is a more serious issue, in that Pr^L is not even locally small – for any $C \in \operatorname{Pr}^L$, one has $\operatorname{Hom}_{\operatorname{Pr}^L}(\operatorname{Ani}, C) = C$ as Ani is the free presentable $(\infty, 1)$ -category on one object.

On the other hand, $\operatorname{Pr}_\kappa^L$ is locally small – for example, $\operatorname{Hom}_{\operatorname{Pr}_\kappa^L}(\operatorname{Ani}, C) = C^\kappa$. So $\operatorname{Pr}_\kappa^L$ can be defined in our framework.

⁸In the quasicategory model for $(\infty, 1)$ -categories, this is a strict 1-category enriched in Kan complexes; one then passes to the homotopy-coherent nerve.

We see that Pr^L is too large to be an object of itself – it is not even locally small. However:

THEOREM 2.9 (Campion [Cam22], [Aok25a, Proposition 2.3]). *The $(\infty, 1)$ -category Pr_κ^L is itself κ -presentable, i.e.*

$$\mathrm{Pr}_\kappa^L \in \mathrm{Pr}_\kappa^L.$$

As written, this seems to contradict the axiom of Foundation of ZFC, but upon closer inspection of our encoding, this disappears. Namely, in our encoding the actual objects of Pr_κ^L are not κ -presentable C but their subcategories C^κ of κ -compact objects.

PROOF. The starting point is “Gabriel–Ulmer duality” (cf. [GU71], [Lur09b, Proposition 5.5.7.8, Proposition 5.5.7.10]), giving an equivalence between Pr_κ^L and the $(\infty, 1)$ -category Rex_κ of small $(\infty, 1)$ -categories D admitting all κ -small colimits and retracts, and functors preserving κ -small colimits.⁹ In one direction, this takes C to C^κ ; in the reverse direction D to $\mathrm{Ind}_\kappa(D)$. Implicitly, we in fact used this equivalence in our definition of Pr_κ^L .

Thus, it is enough to prove that Rex_κ is κ -presentable. First, it is clearly locally small. Now consider the forgetful functor

$$G : \mathrm{Rex}_\kappa \rightarrow \mathrm{Cat}_\infty$$

to the $(\infty, 1)$ -category of small $(\infty, 1)$ -categories. This admits a left adjoint F , taking any $S \in \mathrm{Cat}_\infty$ to $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani})^\kappa$; indeed, this follows from Gabriel–Ulmer duality and $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani})$ being the free presentable $(\infty, 1)$ -category on S , cf. [Lur09b, Theorem 5.1.5.6]. We claim that this yields a monadic adjunction, making Rex_κ equivalent to algebras over the monad GF on Cat_∞ . By Barr–Beck–Lurie [Lur17, Theorem 4.7.3.5], it suffices to show that G is conservative, and Rex_κ admits and G preserves G -split geometric realizations. It is clear that G is conservative. Consider a simplicial diagram $C_\bullet$ in Rex_κ with geometric realization C_{-1} in Cat_∞ , and assume that $C_\bullet \rightarrow C_{-1}$ is split in Cat_∞ . For any κ -small diagram I , also $\mathrm{Fun}(I, C_\bullet) \rightarrow \mathrm{Fun}(I, C_{-1})$ is a (split) geometric realization in Cat_∞ ; applying this to I and $I^\mathcal{P}$, one easily deduces the claim.

As Cat_∞ has all small colimits and Rex_κ is monadic over Cat_∞ , it follows that Rex_κ has all small colimits. Moreover, Cat_∞ is compactly generated (by simplex categories Δ^n), so it suffices to see that F preserves κ -compact objects. This is equivalent to showing that its right adjoint G preserves κ -filtered colimits. This is [Lur09b, Proposition 5.5.7.11]; let us briefly recall the argument. The key is to see that for a κ -filtered diagram D_i , $i \in I$, of objects $D_i \in \mathrm{Rex}_\kappa$, their κ -filtered colimit $D = \mathrm{colim}_i D_i$ in Cat_∞ still admits κ -small colimits, and all functors $D_i \rightarrow D$ preserve κ -small colimits. But any κ -small diagram $J \rightarrow D$ factors over $J \rightarrow D_i$ for some i , as I is κ -filtered. One can then take the colimit along J in D_i , and this colimit is preserved under all further functors $D_i \rightarrow D_j$ (as the functors commute with κ -small colimits); this ensures that that it also defines a colimit in D . $\square$

QUESTION 2.10. Given that Pr_κ^L is κ -presentable, it has a large subcategory $(\mathrm{Pr}_\kappa^L)^\kappa$ of κ -compact objects. Is it, in fact, a κ -compact object of itself, i.e.

$$\mathrm{Pr}_\kappa^L \in (\mathrm{Pr}_\kappa^L)^\kappa?$$

Campion claims that this is false in [Cam22]. This is not quite the full truth:

PROPOSITION 2.11. *One has $\mathrm{Pr}_\kappa^L \in (\mathrm{Pr}_\kappa^L)^\kappa$ if and only if κ is not a limit cardinal.*

⁹Any functor preserves retracts; and if $\kappa > \aleph_0$, then existence of retracts follows from existence of κ -small colimits.

Recall that regular limit cardinals are precisely the weakly inaccessible cardinals, so the condition is that κ is *not* weakly inaccessible. We will be mostly interested in $\kappa = \aleph_1$ which is not a limit cardinal, so Pr_κ^L is a κ -compact object of itself. It seems a bit awkward that the good behaviour happens precisely for κ which are *not* (weakly) inaccessible.

SKETCH. If κ is a limit cardinal and uncountable, then

$$\text{Pr}_\kappa^L = \text{colim}_{\lambda < \kappa} \text{Pr}_\lambda^L$$

in Pr_κ^L , where the colimit diagram is of size κ (so not κ -small), and in fact κ -filtered by regularity of κ , and Pr_κ^L is not a retract of Pr_λ^L for $\lambda < \kappa$. Thus, Pr_κ^L cannot be a κ -compact object of Pr_κ^L . The case $\kappa = \aleph_0$ requires a special argument.

Now assume that κ is not a limit cardinal so $\kappa = \lambda^+$ is the successor of some (infinite) cardinal λ . First, we note that Cat_∞ is a κ -compact object of $\text{Pr}_{\aleph_0}^L$, and hence in Pr_κ^L . Now to ensure the existence and preservation of general κ -small colimits, we only need to consider initial objects, pushouts, and disjoint unions of λ many objects. This describes Rex_κ as objects $D \in \text{Cat}_\infty$ together with an object $\emptyset \in D$, a pushout operation $\text{Fun}(\Delta^1 \sqcup_{\Delta^0} \Delta^1, C) \rightarrow C$ and a disjoint union operation $D^\lambda \rightarrow D$ satisfying certain compatibilities (countably many). From this description, one can see that Rex_κ is a κ -compact object of Pr_κ^L . $\square$

Using this κ -compactness (assuming κ is a successor cardinal), we can formulate a more finitary version of a “category of categories that contains itself”, running into no class-sized categories at all. Namely, we can consider $\text{Rex}_\kappa^\kappa = (\text{Rex}_\kappa)^\kappa$, the (essentially) small $(\infty, 1)$ -category of κ -compact objects in the κ -presentable $(\infty, 1)$ -category Rex_κ (of small $(\infty, 1)$ -categories admitting κ -small colimits, with functors preserving κ -small colimits). In other words, the objects of Rex_κ^κ are those small $(\infty, 1)$ -categories with κ -small colimits that admit some κ -small presentation.

COROLLARY 2.12. *For successor cardinals κ , we have $\text{Rex}_\kappa^\kappa \in \text{Rex}_\kappa^\kappa$.*

Here, the apparent contradiction to the Foundation axiom of ZFC is only resolved by closely inspecting how one has, implicitly, turned essentially small categories into actually small categories.

PROOF. Under the equivalence $\text{Pr}_\kappa^L \cong \text{Rex}_\kappa$, the object Pr_κ^L is mapped to $(\text{Pr}_\kappa^L)^\kappa$. But this is equivalent to Rex_κ^κ , and by the previous proposition is a κ -compact object. $\square$

To a large extent, the constructions of the following lectures could be executed in either the setting of $\text{Pr}_\kappa^L \in \text{Pr}_\kappa^L$ or $\text{Rex}_\kappa^\kappa \in \text{Rex}_\kappa^\kappa$, and actually the two settings will lead to equivalent constructions (up to applying Gabriel–Ulmer duality). We prefer to work in the setting of Pr_κ^L because we will often use the adjoint functor theorem.

3. Lecture III: Algebra with Categories

Lurie's works [Lur09b] and [Lur17] give a very convenient framework to do algebra in (symmetric monoidal) $(\infty, 1)$ -categories, such as the category $D(\mathbb{S})$ of spectra (in the sense of algebraic topology) – leading to the notions of E_∞ -algebras, their $(\infty, 1)$ -categories of modules, etc. But they also give something even more profound: Namely, all the ideas can be applied one categorical level higher to the symmetric monoidal $(\infty, 1)$ -category Pr^L , making it possible to do algebra with categories. It turns out that this is an extraordinarily powerful idea whose full power is only slowly being realized; it also shifts categories from being a meta-mathematical organizing tool to being the objects that are manipulated.

The first important result about presentable $(\infty, 1)$ -categories is the adjoint functor theorem.

THEOREM 3.1 ([Lur09b, Corollary 5.5.2.9]). *Let $F : C \rightarrow D$ be a functor between presentable $(\infty, 1)$ -categories.*

- (1) *The functor F admits a right adjoint if and only if F commutes with all small colimits.*
- (2) *The functor F admits a left adjoint if and only if F commutes with all small limits, and there is some regular cardinal κ such that F commutes with κ -filtered colimits.*

The condition that there is some κ for which F commutes with κ -filtered colimits is also known as “ F is accessible”. For a given κ , it is equivalent to the condition that the left adjoint F^L of F preserves κ -compact objects.

Together with the functoriality of passing to adjoint, this yields the following corollary. (As stated, this requires us to pass to a larger universe; the version with κ can be stated without.)

COROLLARY 3.2 ([Lur09b, Corollary 5.5.3.4]). *Let Pr^R be the $(\infty, 1)$ -category whose objects are presentable $(\infty, 1)$ -categories, and morphisms are accessible functors that commute with all small limits. Then there is an equivalence*

$$(\mathrm{Pr}^L)^{\mathrm{op}} \cong \mathrm{Pr}^R$$

given by passing to adjoint functors.

Similarly, let Pr_κ^R be the $(\infty, 1)$ -category whose objects are κ -presentable $(\infty, 1)$ -categories, and morphisms are functors that commute with κ -filtered colimits, and all small limits. Then the above equivalence restricts to an equivalence

$$(\mathrm{Pr}_\kappa^L)^{\mathrm{op}} \cong \mathrm{Pr}_\kappa^R.$$

Let us now discuss various operations within Pr^L .

3.1. Limits of presentable $(\infty, 1)$ -categories. The simplest operation to understand are limits.

PROPOSITION 3.3. *The $(\infty, 1)$ -category Pr^L admits all small limits, and the forgetful functor to (large) $(\infty, 1)$ -categories preserves small limits.*

Similarly, the $(\infty, 1)$ -category Pr_κ^L admits all small limits, and the composite functor $\mathrm{Pr}_\kappa^L \cong \mathrm{Rex}_\kappa \rightarrow \mathrm{Cat}_\infty$ preserves small limits.

If κ is uncountable, the inclusion $\mathrm{Pr}_\kappa^L \rightarrow \mathrm{Pr}^L$ preserves κ -small limits.

We will generally take $\kappa = \aleph_1$; in this case, countable limits taken in Pr_κ^L and in Pr^L agree, and also agree with the naive limit of $(\infty, 1)$ -categories.

PROOF. The first part is [Lur09b, Proposition 5.5.3.13]. For the second part, it suffices to see that $\text{Rex}_\kappa \rightarrow \text{Cat}_\infty$ preserves small limits. For this, let $I \rightarrow \text{Rex}_\kappa : i \mapsto C_i$ be some diagram in Rex_κ , and let $C = \lim_i C_i$ be the limit in Cat_∞ . Then C admits κ -small colimits, and these commute with the κ -small colimits in the C_i – this easily follows from all transition functors commuting with κ -small colimits which ensures that taking colimits termwise in each C_i actually produces an object in C again, which is then the desired colimit in C . This easily implies that C is in fact the limit in Rex_κ .

For the final statement, it suffices to handle κ -small coproducts, and fibre products. If I is κ -small and $C_i \in \text{Pr}_\kappa^L$, then we need to see that the natural functor

$$\text{Ind}_\kappa(\prod_i C_i^\kappa) \rightarrow \prod_i C_i$$

is an equivalence. First, $\prod_i C_i^\kappa \subset (\prod_i C_i)^\kappa$ as each object of $\prod_i C_i^\kappa$ is the κ -small coproduct of its coordinates, and hence a κ -small colimit of κ -compact objects, and so itself κ -compact. On the other hand, each object of $\prod_i C_i$ is the coproduct of its coordinates, and those are colimits of objects in C_i^κ , ensuring that every object of $\prod_i C_i$ is a colimit of objects in $\prod_i C_i^\kappa$. As $\prod_i C_i^\kappa \subset \prod_i C_i$ is stable under κ -small colimits, the equivalence follows.

In the case of fibre products, we need to see that for uncountable κ , a fibre product $D \times_C E$ of κ -small presentable $(\infty, 1)$ -categories along functors preserving κ -compact objects is κ -presentable, and the resulting functors preserve κ -compact objects. We recall the argument from [Aok25a, Proposition 3.2]. We have to see that the natural functor

$$\text{Ind}_\kappa(D^\kappa \times_{C^\kappa} E^\kappa) \rightarrow D \times_C E$$

is an equivalence. It is clear that $D^\kappa \times_{C^\kappa} E^\kappa$ consists of κ -compact objects in $D \times_C E$, so the functor is fully faithful; also, $D^\kappa \times_{C^\kappa} E^\kappa$ is stable under κ -small colimits. It remains to see that any object $X \in D \times_C E$ is a colimit of objects in $D^\kappa \times_{C^\kappa} E^\kappa$. Consider the images $Y \in D$, $Z \in E$, $W \in C$ of $X \in D \times_C E$. We get a cartesian diagram

$$\begin{array}{ccc} (D^\kappa \times_{C^\kappa} E^\kappa)_{/X} & \longrightarrow & E_{/Z}^\kappa \\ \downarrow & & \downarrow \\ D_{/Y}^\kappa & \longrightarrow & C_{/W}^\kappa \end{array}$$

Here all corners except the upper left are known to be κ -filtered, and all functors preserve κ -small colimits. Moreover, the right and bottom maps are cofinal. It suffices to see that also the upper left term is κ -filtered, and the left and top maps are cofinal: Then the colimit of the tautological diagram

$$(D^\kappa \times_{C^\kappa} E^\kappa)_{/X} \rightarrow D \times_C E$$

is given by $X \in D \times_C E$ (as the natural map to X becomes an isomorphism in D and E). This cofinality follows from [Lur09b, Lemma 5.4.6.5] (applied with $\tau := \kappa$, $\kappa := \aleph_0$). $\square$

3.2. Colimits of presentable $(\infty, 1)$ -categories. Colimits of presentable $(\infty, 1)$ -categories turn out to be more interesting – namely, as we will see, they are very far from the naive colimits of $(\infty, 1)$ -categories. The naive colimit usually fails to have all small colimits, so these have to be adjoined again. It is at first not clear that this can be done in a controllable way. Fortunately, the adjoint functor theorem gives a way to understand the situation:

THEOREM 3.4 ([**Lur09b**, Theorem 5.5.3.18, Proposition 5.5.7.6]). *The $(\infty, 1)$ -category Pr^L admits all small colimits. Equivalently, the $(\infty, 1)$ -category Pr^R admits all small limits; moreover, they commute with the forgetful functor to $(\infty, 1)$ -categories.*

Similarly, Pr_κ^L admits all small colimits, and the inclusion $\mathrm{Pr}_\kappa^L \rightarrow \mathrm{Pr}^L$ commutes with all small colimits.

The key content here is that the limit along the right adjoint functors taken in $(\infty, 1)$ -categories turns out to be presentable (and all relevant functors accessible); once one knows this, it is easy to see that it satisfies the desired universal property.

Let us compute some examples.

EXAMPLE 3.5.

- (1) Let I be a set, and let $C_i \in \mathrm{Pr}^L$ be presentable $(\infty, 1)$ -categories. Then the product of the C_i in Pr^L agrees with the naive product $\prod_i C_i$ as $(\infty, 1)$ -categories.
- (1 $_\kappa$) Let I be a set, and let $C_i \in \mathrm{Pr}_\kappa^L$ be κ -presentable $(\infty, 1)$ -categories. Then the product of the C_i in Pr_κ^L is given by

$$\prod_i^{\mathrm{Pr}_\kappa^L} C_i = \mathrm{Ind}_\kappa(\prod_i C_i^\kappa).$$

There is a natural functor $\prod_i^{\mathrm{Pr}_\kappa^L} C_i \rightarrow \prod_i C_i$ which is an equivalence of $|I| < \kappa$.

- (2) Let I be a set, and let $C_i \in \mathrm{Pr}^L$ be presentable $(\infty, 1)$ -categories. Then the coproduct of the C_i in Pr^L also agrees with the naive product $\prod_i C_i$ as $(\infty, 1)$ -categories. Indeed, it agrees with limit along the right adjoint functors as transition maps; but for a product, there are no transition maps, so we just get the product. In particular, for any set I , the natural map

$$\bigsqcup_i^{\mathrm{Pr}^L} C_i \rightarrow \prod_i^{\mathrm{Pr}^L} C_i = \prod_i C_i$$

is an isomorphism. In other words (adapting the notion of semiadditivity from [**Lur17**, Definition 6.1.6.13] to arbitrary cardinals), it is κ -semiadditive for all cardinals κ . This is an extremely unusual property!

- (2 $_\kappa$) Let I be a set, and let $C_i \in \mathrm{Pr}_\kappa^L$ be κ -presentable $(\infty, 1)$ -categories. Then the coproduct of the C_i in Pr_κ^L is given by

$$\bigsqcup_i^{\mathrm{Pr}_\kappa^L} C_i = \bigsqcup_i^{\mathrm{Pr}^L} C_i \cong \prod_i^{\mathrm{Pr}^L} C_i = \prod_i C_i.$$

Thus, slightly confusingly, the coproduct of the C_i (in Pr_κ^L) is always the naive product, while the product of the C_i (in Pr_κ^L) is in general different. However, if $|I| < \kappa$, they agree, so Pr_κ^L is κ -semiadditive.

- (3) Let S be a small $(\infty, 1)$ -category. Then the free presentable $(\infty, 1)$ -category $\langle S \rangle_{\mathrm{pres}}$ with a functor from S is given by $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani})$, cf. [**Lur09b**, Theorem 5.1.5.6, Remark 5.5.3.7]. This can also be seen as an instance of colimits (or free constructions) in Pr^L agreeing with limits (or cofree constructions). Indeed, given any presentable C with a functor $i : S \rightarrow C$, we get a functor $C \rightarrow \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani})$ via $X \mapsto (T \in S \mapsto \mathrm{Hom}_C(i(T), X))$. This functor

commutes with all limits and is accessible, thus admits a left adjoint $\text{Fun}(S^{\text{op}}, \text{Ani}) \rightarrow C$, extending the given functor i , by construction. We note that $\langle S \rangle_{\text{pres}} = \text{Fun}(S^{\text{op}}, \text{Ani})$ is compactly generated by S , in particular lies in Pr_{κ}^L for any κ .

- (4) Let S be a small $(\infty, 1)$ -category and let R be a set of maps in $\langle S \rangle_{\text{pres}} = \text{Fun}(S^{\text{op}}, \text{Ani})$. Then there is a free presentable $(\infty, 1)$ -category generated by S in which all maps in R go to isomorphisms,

$$\langle S \rangle_{\text{pres}}[R^{-1}]_{\text{pres}}.$$

Concretely, this is given by the full subcategory

$$\text{Fun}^R(S^{\text{op}}, \text{Ani}) \subset \text{Fun}(S^{\text{op}}, \text{Ani})$$

of all functors $F : S^{\text{op}} \rightarrow \text{Ani}$ such that for all maps $G \rightarrow G'$ in R , the map $\text{Hom}(G', F) \rightarrow \text{Hom}(G, F)$ is an isomorphism. Indeed, the functor

$$\langle S \rangle_{\text{pres}} = \text{Fun}(S^{\text{op}}, \text{Ani}) \rightarrow \langle S \rangle_{\text{pres}}[R^{-1}]_{\text{pres}}$$

commutes with all colimits and hence has a right adjoint. This right adjoint is automatically fully faithful, identifying $\langle S \rangle_{\text{pres}}[R^{-1}]_{\text{pres}}$ with the full subcategory $\text{Fun}^R(S^{\text{op}}, \text{Ani})$ described above, cf. [Lur09b, Proposition 5.5.4.20]. This is then another instance of describing free constructions in Pr^L as cofree constructions, in terms of passing to right adjoints.

- (5) The functor $S \mapsto \langle S \rangle_{\text{pres}}$ from Cat_{∞} to Pr^L commutes with all small colimits. As a particular instance, consider the case of pushouts of classifying spaces of groups $*/G \leftarrow * \rightarrow */H$. The pushout in Cat_{∞} gives $*/(G * H)$ where $G * H$ is the free product of G and H . This is a group that is not in general easily describable. The identification of colimits in Pr^L with limits says in this case

$$\text{Fun}(*/(G * H)^{\text{op}}, \text{Ani}) \cong \text{Fun}(* / G^{\text{op}}, \text{Ani}) \times_{\text{Ani}} \text{Fun}(* / H^{\text{op}}, \text{Ani}),$$

giving a concise description of the free presentable $(\infty, 1)$ -category on $*/(G * H)$. It is saying in this case that to give an action of $G * H$ is simply to give actions of G and H individually. This is a general phenomenon: Colimits in Cat_{∞} give complicated free constructions, but their presentable versions admit compact descriptions by using right adjoints.

- (6) Any presentable $(\infty, 1)$ -category C admits a presentation as in (4). For example, if C is κ -presentable, one can take $S = C^{\kappa}$ and R to consist of all maps from a κ -small colimit of objects of C^{κ} taken in $\text{Fun}((C^{\kappa})^{\text{op}}, \text{Ani})$ to the κ -small colimit taken in C^{κ} . Then

$$C = \text{Ind}_{\kappa}(C^{\kappa}) \cong \text{Fun}^{\kappa\text{-lex}}((C^{\kappa})^{\text{op}}, \text{Ani})$$

consists of functors $(C^{\kappa})^{\text{op}} \rightarrow \text{Ani}$ commuting with κ -small colimits. In particular, in this case R consists of maps between κ -small objects of $\langle S \rangle_{\text{pres}}$. Conversely, if R consists of maps between κ -small objects, then S maps to κ -compact objects of C and hence C is κ -presentable.

3.3. Tensor products of presentable $(\infty, 1)$ -categories. Now we can also discuss tensor products.

DEFINITION/PROPOSITION 3.6. *Let C and D be presentable $(\infty, 1)$ -categories. There is a universal presentable $(\infty, 1)$ -category $C \otimes D$ equipped with a functor*

$$C \times D \rightarrow C \otimes D$$

that commutes with small colimits in each variable. If C and D are κ -presentable, then $C \otimes D$ is κ -presentable, and $C \times D \rightarrow C \otimes D$ preserves κ -compact objects. This endows Pr^L and Pr_κ^L with compatible symmetric monoidal structures, with unit object Ani .

To properly define the symmetric monoidal structure, we should phrase the universal property for any finite set of presentable $(\infty, 1)$ -categories.

PROOF. (See [Lur17, Proposition 4.8.1.15].) It suffices to prove existence. For this, choose presentations

$$C = \langle S \rangle_{\mathrm{pres}}[R^{-1}]_{\mathrm{pres}}, \quad D = \langle T \rangle_{\mathrm{pres}}[Q^{-1}]_{\mathrm{pres}}.$$

Then it is easy to see that

$$C \otimes D := \langle S \times T \rangle_{\mathrm{pres}}[(R \times Q)^{-1}]_{\mathrm{pres}}$$

has the desired universal property. If C and D are κ -presentable, we can assume that $S = C^\kappa$ and $T = D^\kappa$, and R and Q consist of maps between κ -compact objects, in which case the same is true for $R \times Q$; hence $S \times T = C^\kappa \times D^\kappa$ maps to κ -compact objects of $C \otimes D$. $\square$

EXAMPLE 3.7. Let us compute several tensor products.

- (1) Assume $C = \langle S \rangle_{\mathrm{pres}} = \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani})$. Then

$$C \otimes D = \mathrm{Fun}(S^{\mathrm{op}}, D).$$

Indeed, for general C as in the proof, we have

$$\begin{aligned} C \otimes D &= \mathrm{Fun}^{R \times Q}(S^{\mathrm{op}} \times T^{\mathrm{op}}, \mathrm{Ani}) \\ &= \mathrm{Fun}^R(S^{\mathrm{op}}, \mathrm{Fun}^Q(T^{\mathrm{op}}, \mathrm{Ani})) \\ &= \mathrm{Fun}^R(S^{\mathrm{op}}, D). \end{aligned}$$

When $R = \emptyset$, this gives the above example.

- (2) In general, the previous computation shows that when C is κ -presentable, we have

$$C \otimes D = \mathrm{Fun}^{\kappa\text{-lex}}((C^\kappa)^{\mathrm{op}}, D).$$

- (3) Let A be a ring. Then $D_{\geq 0}(A)$ has compact projective generators A^n for $n \geq 0$, and so

$$D_{\geq 0}(A) = \mathrm{Fun}^{\mathrm{finite prod}}(\{A^n\}^{\mathrm{op}}, \mathrm{Ani}).$$

It follows that for any D , we have

$$D_{\geq 0}(A) \otimes D = \mathrm{Fun}^{\mathrm{finite prod}}(\{A^n\}^{\mathrm{op}}, D).$$

This can be understood as “ A -module objects in D ”.

Adjoint to the tensor product, we get an internal Hom functor. This is in fact just recovering the obvious functor categories:

PROPOSITION 3.8. *The symmetric monoidal $(\infty, 1)$ -category Pr^L is cartesian closed. For $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L$, the internal Hom is given by the presentable $(\infty, 1)$ -category $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ of colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$.*

Similarly, the presentable symmetric monoidal $(\infty, 1)$ -category Pr_κ^L is cartesian closed. For $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}_\kappa^L$, the internal Hom in Pr_κ^L is given by the κ -presentable $(\infty, 1)$ -category

$$\mathrm{Ind}_\kappa \mathrm{Fun}^{\mathrm{Rex}_\kappa}(\mathcal{C}^\kappa, \mathcal{D}^\kappa).$$

If $\mathcal{C} \in \mathrm{Pr}_\kappa^L$ is a κ -compact object, then the natural comparison map

$$\mathrm{Ind}_\kappa \mathrm{Fun}^{\mathrm{Rex}_\kappa}(\mathcal{C}^\kappa, \mathcal{D}^\kappa) \rightarrow \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}),$$

between the two versions of the internal Hom, is an isomorphism.

PROOF. For the first part, it is clear that $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ has the correct universal property, once one knows that it is presentable; this is [Lur09b,]. Similarly, in Pr_κ^L , the description follows from Gabriel–Ulmer duality. It remains to see that if $\mathcal{C}$ is a κ -compact object of Pr_κ^L , the two descriptions agree. As κ -small limits are the same in Pr_κ^L and Pr^L , we can assume that $\mathcal{C}$ is the free presentable $(\infty, 1)$ -category on a simplex Δ^n . In this case, it says that

$$\mathrm{Ind}_\kappa \mathrm{Fun}(\Delta^n, \mathcal{D}^\kappa) \rightarrow \mathrm{Fun}(\Delta^n, \mathcal{D})$$

is an equivalence for all κ -presentable $\mathcal{D}$, which is just the statement that the Δ^n are κ -compact in Pr_κ^L . $\square$

Now let C be any symmetric monoidal κ -presentable $(\infty, 1)$ -category; we always assume here that the tensor product of C commutes with colimits in each variable and preserves κ -compact objects (and the unit of C is κ -compact). In other words, this is a commutative algebra object in $(\mathrm{Pr}_\kappa^L, \otimes)$. For any algebra $A \in \mathrm{Alg}(C)$, one can define the κ -presentable $(\infty, 1)$ -category of (left) modules of A in C ,

$$\mathrm{Mod}_A(C),$$

cf. [Lur17, Section 4.2]. These are, intuitively, objects $M \in C$ together with an action map $A \otimes_C M \rightarrow M$ satisfying unitality and associativity as expressed in (higher) coherences. The forgetful functor to C has a left adjoint $N \mapsto N \otimes_C A$ of “free A -modules”. If A is commutative, then $\mathrm{Mod}_A(C)$ acquires a symmetric monoidal structure making $C \rightarrow \mathrm{Mod}_A(C)$ symmetric monoidal, cf. [Lur17, Section 4.5]. This tensor product is given by

$$(M, M') \mapsto M \otimes_A M' = \mathrm{colim}(\dots \rightarrow M \otimes_C A \otimes_C M' \rightarrow M \otimes_C M')$$

the geometric realization of the simplicial Bar complex. This is a countable colimit, and it becomes important that such countable colimits exist, and are preserved by various constructions. The occurrence of such countable colimits in the basic theory of modules is a key reason that we need countable colimits and hence choose $\kappa = \aleph_1$.

Commutative algebra objects B in $\mathrm{Mod}_A(C)$ are equivalent to commutative algebra objects in C equipped with a map from A ,

$$\mathrm{CAlg}(C)_{A/} \cong \mathrm{CAlg}(\mathrm{Mod}_A(C)).$$

Now $\mathrm{Mod}_A(C)$ is itself a symmetric monoidal κ -presentable $(\infty, 1)$ -category over C . Applying the previous displayed identification for $C = \mathrm{Pr}_\kappa^L$ and $A = C$, we thus get

$$\mathrm{Mod}_A(C) \in \mathrm{CAlg}(\mathrm{Pr}_\kappa^L)_{C/} \cong \mathrm{CAlg}(\mathrm{Mod}_C(\mathrm{Pr}_\kappa^L)).$$

DEFINITION 3.9. Let C be a symmetric monoidal κ -presentable $(\infty, 1)$ -category. The symmetric monoidal $(\infty, 1)$ -category of C -linear κ -presentable $(\infty, 1)$ -categories is

$$\mathrm{Mod}_C(\mathrm{Pr}_\kappa^L).$$

We note $\mathrm{Mod}_C(\mathrm{Pr}_\kappa^L)$ is, itself, symmetric monoidal κ -presentable, by applying the above discussion to $C = \mathrm{Pr}_\kappa^L$ (which is symmetric monoidal κ -presentable) and $A = C$.

REMARK 3.10. If $D \in \mathrm{Mod}_C(\mathrm{Pr}_\kappa^L)$, then for any $X, Y \in D$ one has an object $\mathrm{Hom}_D(X, Y) \in C$, determined as the universal object of C with a map $\mathrm{Hom}_D(X, Y) \otimes X \rightarrow Y$ (using the action $C \times D \rightarrow C \otimes D \rightarrow D$). This determines an enrichment of D in C . We refer to [Hei23], [GH15], [BM24], [HM24, Appendix C], [RZ25] for a discussion of enriched ∞ -categories and their relation to the presentable theory used here.

THEOREM 3.11. *The functor $A \mapsto \mathrm{Mod}_A(C)$ yields a fully faithful functor*

$$\mathrm{CAlg}(C) \hookrightarrow \mathrm{CAlg}(\mathrm{Mod}_C(\mathrm{Pr}_\kappa^L))$$

whose right adjoint is given by $D \mapsto \mathrm{End}_D(1)$. The displayed map is a map in $\mathrm{CAlg}(\mathrm{Pr}_\kappa^L)$.

PROOF. The fully faithfulness is [Lur17, Corollary 4.8.5.21]; the identification of the right adjoint is [Lur17, Remark 4.8.5.12]. That everything lives in $\mathrm{CAlg}(\mathrm{Pr}_\kappa^L)$ is checked in [Aok25a, Lemma 2.11], by checking that the right adjoint $D \mapsto \mathrm{End}_D(1)$ commutes with κ -filtered colimits. $\square$

EXAMPLE 3.12. Let $C = D(A)$ for some commutative ring A . Then it follows that for any commutative A -algebras B and C , one has

$$D(B) \otimes_{D(A)} D(C) = D(B \otimes_A C).$$

Indeed, this is a special case of the commutation of the above functor with colimits. More generally, the theorem says that the algebraic operations of colimits and tensor products as performed on the level of categories faithfully reflect the same operations at the level of rings and modules.

At this point, we have succeeded in pushing all the important algebraic operations to the level of categories. But now we can use the self-referentiality of the previous lecture to blast through all further categorical levels:

Apply Theorem 3.11 to $C = \mathrm{Pr}_\kappa^L$. We get an embedding

$$\mathrm{CAlg}(\mathrm{Pr}_\kappa^L) \hookrightarrow \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Pr}_\kappa^L}(\mathrm{Pr}_\kappa^L)) : C \mapsto \mathrm{Mod}_C(\mathrm{Pr}_\kappa^L).$$

If we define $2\mathrm{Pr}_\kappa^L = \mathrm{Mod}_{\mathrm{Pr}_\kappa^L}(\mathrm{Pr}_\kappa^L)$ (which, like for any C , is a symmetric monoidal κ -presentable $(\infty, 1)$ -category), we thus get a full embedding

$$\mathrm{CAlg}(\mathrm{Pr}_\kappa^L) \hookrightarrow \mathrm{CAlg}(2\mathrm{Pr}_\kappa^L).$$

But then we can apply Theorem 3.11 again to $C = 2\mathrm{Pr}_\kappa^L$. Inductively setting

$$(n+1)\mathrm{Pr}_\kappa^L = \mathrm{Mod}_{n\mathrm{Pr}_\kappa^L}(\mathrm{Pr}_\kappa^L),$$

each of which is itself a symmetric monoidal κ -presentable $(\infty, 1)$ -category, we get fully faithful embeddings

$$\mathrm{CAlg}(n\mathrm{Pr}_\kappa^L) \hookrightarrow \mathrm{CAlg}((n+1)\mathrm{Pr}_\kappa^L) : C \mapsto \mathrm{Mod}_C(n\mathrm{Pr}_\kappa^L).$$

Finally, we can give the precise definition of Stefanich rings. From now on, we take $\kappa = \aleph_1$ and abbreviate $n\mathrm{Pr} = n\mathrm{Pr}_{\aleph_1}^L$.

DEFINITION 3.13. The symmetric monoidal κ -presentable $(\infty, 1)$ -category of Stefanich rings is

$$\begin{aligned} \text{StRing} &= \text{colim}_{\text{pres}}(\text{CAlg}(1\text{Pr}) \hookrightarrow \text{CAlg}(2\text{Pr}) \hookrightarrow \dots) \\ &= \lim_{\text{End}_-(1)}(\text{CAlg}(1\text{Pr}) \leftarrow \text{CAlg}(2\text{Pr}) \leftarrow \dots); \end{aligned}$$

that is, of sequences $(A_1, A_2, \dots)$ where $A_n \in \text{CAlg}(n\text{Pr})$ with isomorphisms

$$A_n \cong \text{End}_{A_{n+1}}(1).$$

REMARK 3.14. The embedding

$$\text{CAlg}(1\text{Pr}) \hookrightarrow \text{StRing}$$

that is obvious from the colimit description is in the limit description given by

$$C \mapsto (C, 1\text{Pr}_C := \text{Mod}_C(1\text{Pr}), 2\text{Pr}_C := \text{Mod}_{1\text{Pr}_C}(2\text{Pr}), \dots).$$

REMARK 3.15. We are consistently ignoring (∞, n) -categorical structures when $n > 1$, treating $n\text{Pr}$ merely as an $(\infty, 1)$ -category. We could remember its structure as an $(\infty, n+1)$ -category, using which also $\text{CAlg}(n\text{Pr})$ has an $(\infty, n+1)$ -category structure. However, the transition functors defining StRing are incompatible with the higher morphisms, so that StRing really only is an $(\infty, 1)$ -category.

REMARK 3.16. As everything is κ -compactly generated, it can be equivalently described through the κ -compact objects. This in fact gives a description in terms of the self-referential $\text{Rex}_\kappa^\kappa \in \text{Rex}_\kappa^\kappa$. More precisely, if we inductively define $1\text{Rex}_\kappa^\kappa = \text{Rex}_\kappa^\kappa$ and

$$(n+1)\text{Rex}_\kappa^\kappa = \text{Mod}_{n\text{Rex}_\kappa^\kappa}(\text{Rex}_\kappa^\kappa) \in \text{CAlg}(\text{Rex}_\kappa^\kappa),$$

then $(n\text{Pr}_\kappa^L)^\kappa = \text{Ind}_\kappa(n\text{Rex}_\kappa^\kappa)$; i.e., $n\text{Pr}_\kappa^L = n\text{Rex}_\kappa$ if we formally define $n\text{Rex}_\kappa = \text{Ind}_\kappa(n\text{Rex}_\kappa^\kappa)$ (leading to consistent notation). This describes

$$\text{StRing}^\kappa = \text{colim}_{\text{Rex}_\kappa}(\text{CAlg}(\text{Rex}_\kappa^\kappa) \hookrightarrow \text{CAlg}(2\text{Rex}_\kappa^\kappa) \hookrightarrow \dots) \in \text{CAlg}(\text{Rex}_\kappa^\kappa).$$

(Note that in this case, we do not have a description in terms of a limit along right adjoints, as the right adjoint functors $D \mapsto \text{End}_D(1)$ do not preserve κ -compact objects.) This gives a description of StRing in more “finitary” terms – this description does not involve any class-sized categories.

4. Lecture IV: Quasicoherent Sheaves of Higher Categories, I

The goal of this lecture and the next is to associate to any algebraic stack X a Stefanich ring

$$(\mathcal{O}(X), D(X), \mathrm{Pr}_X, 2\mathrm{Pr}_X, \dots)$$

and study the properties of the resulting associations $X \mapsto n\mathrm{Pr}_X$ — in fact, they will all be 6-functor formalisms, with interesting properties.

An extremely brief history: The case $X \mapsto D(X)$ is of course classical and goes back to the work of the Grothendieck school on coherent duality. One categorical level up, the case $X \mapsto \mathrm{Pr}_X$ was studied notably by Gaitsgory [Gai15] and Lurie. The case $n \geq 2$ was developed in Stefanich's thesis [Ste21].

4.1. Higher linear categories. In this lecture, we start with the affine case. Let C be (the opposite of) some $(\infty, 1)$ -category of derived rings; this could be animated rings, connective $\mathbb{E}_\infty$ - $\mathbb{Z}$ -algebras, connective $\mathbb{E}_\infty$ - $\mathbb{S}$ -algebras, or even the whole $(\infty, 1)$ -category of $\mathbb{E}_\infty$ -algebras in $D(\mathbb{S})$. For definiteness, let us take $\mathbb{E}_\infty$ -algebras in $D(\mathbb{S})$. We formally denote objects of C as $\mathrm{Spec}(A)$.

Thus, the preceding lecture gives us a fully faithful, colimit-preserving functor

$$C^{\mathrm{op}} \rightarrow \mathrm{StRing} : \mathrm{Spec}(A) \mapsto (D(A), 1\mathrm{Pr}_A, 2\mathrm{Pr}_A, \dots).$$

Here, inductively $(n+1)\mathrm{Pr}_A = \mathrm{Mod}_{n\mathrm{Pr}_A}((n+1)\mathrm{Pr})$. Note that in particular each $n\mathrm{Pr}_A \in \mathrm{CAlg}((n+1)\mathrm{Pr})$. But $(n+1)\mathrm{Pr} = \mathrm{Mod}_{n\mathrm{Pr}}(1\mathrm{Pr})$, and in general for any symmetric monoidal presentable $(\infty, 1)$ -category C (here, $1\mathrm{Pr}$) with a commutative algebra object $B \in \mathrm{CAlg}(C)$ (here, $n\mathrm{Pr}$), one has $\mathrm{CAlg}(\mathrm{Mod}_B(C)) \cong \mathrm{CAlg}(C)_{B/}$. Moreover, this equivalence is compatible with passing to module categories. Applying this to the object

$$n\mathrm{Pr}_A \in \mathrm{CAlg}((n+1)\mathrm{Pr}) = \mathrm{CAlg}(\mathrm{Mod}_{n\mathrm{Pr}}(1\mathrm{Pr})) = \mathrm{CAlg}(1\mathrm{Pr})_{n\mathrm{Pr}/},$$

we see that we can also write

$$(n+1)\mathrm{Pr}_A = \mathrm{Mod}_{n\mathrm{Pr}_A}(1\mathrm{Pr}),$$

giving an inductive definition of $n\mathrm{Pr}_A \in \mathrm{CAlg}(1\mathrm{Pr})$. In particular, all of these higher categories are just instances of $\mathcal{V}$ -linear categories $\mathrm{Mod}_{\mathcal{V}}(1\mathrm{Pr})$, for some $\mathcal{V} \in \mathrm{CAlg}(1\mathrm{Pr})$.

For any $n \geq 0$, we get a functor

$$C^{\mathrm{op}} \rightarrow \mathrm{CAlg}(1\mathrm{Pr}) : \mathrm{Spec}(A) \mapsto n\mathrm{Pr}_A,$$

where we abusively denote $0\mathrm{Pr}_A := D(A)$. This encodes pullback and tensor functoriality, which by presentability automatically admit right adjoints; thus, we have the four functors $(f^*, f_*, \otimes, \mathrm{Hom})$. We would like to extend this to a 6-functor formalism. For $n = 0$, one option is to declare all maps to be proper, so that $f_! = f_*$; to see that this is sensible, one has to prove that f_* commutes with base change, satisfies the projection formula, and commutes with colimits. In fact, this is true for all $n \geq 0$:

PROPOSITION 4.1. *Let $f : Y = \mathrm{Spec}(B) \rightarrow X = \mathrm{Spec}(A)$ be a map in C . Then*

$$f_n^* : n\mathrm{Pr}_A \rightarrow n\mathrm{Pr}_B$$

admits an $n\mathrm{Pr}_A$ -linear colimit-preserving right adjoint

$$f_{n*} : n\mathrm{Pr}_B \rightarrow n\mathrm{Pr}_A.$$

Moreover, f_{n} commutes with base change.*

If B is a countably presented A -algebra, then f_{n*} also preserves $\aleph_1$ -compact objects, and hence defines a morphism in 1Pr , and hence in $(n+1)\text{Pr}_A = \text{Mod}_{n\text{Pr}_A}(1\text{Pr})$.

If B is not countably presented, f_{n*} would only define a morphism in $\text{Mod}_{n\text{Pr}_A}(\text{Pr}^L)$.

PROOF. Assume $n \geq 1$ (although essentially the same argument works for $n = 0$). Recall that f_n^* is obtained from the map

$$f_{n-1}^* : (n-1)\text{Pr}_A \rightarrow (n-1)\text{Pr}_B \in \text{CAlg}(n\text{Pr})$$

by applying $\text{Mod}_-(n\text{Pr})$. Now more generally, for any $\mathcal{V} \in \text{CAlg}(\text{Pr}^L)$ (here, $n\text{Pr}$) and any map $g : R \rightarrow S$ in $\text{CAlg}(\mathcal{V})$, the symmetric monoidal functor

$$- \otimes_R S : \text{Mod}_R(\mathcal{V}) \rightarrow \text{Mod}_S(\mathcal{V})$$

has a $\mathcal{V}$ -linear colimit-preserving right adjoint, namely the forgetful functor $\text{Mod}_S(\mathcal{V}) \rightarrow \text{Mod}_R(\mathcal{V})$ (interpreting adjoint functors in terms of the corresponding biCartesian fibrations, this follows from [Lur17, Corollary 4.2.3.2, Corollary 4.2.3.5, Theorem 4.5.3.1]). Moreover, for any symmetric monoidal functor $\mathcal{V} \rightarrow \mathcal{V}'$, giving a map $R' \rightarrow S'$ in $\text{CAlg}(\mathcal{V}')$, these functors commute with base change along $\mathcal{V} \rightarrow \mathcal{V}'$.

If $A \rightarrow B$ is countably presented, then inductively $n\text{Pr}_B$ is countably presented over $n\text{Pr}_A$ (as a commutative algebra, or equivalently as a module resp. as an associative algebra); thus, the forgetful functor $n\text{Pr}_B \rightarrow n\text{Pr}_A$ preserves $\aleph_1$ -compact objects — it is enough to check that the image of the unit is $\aleph_1$ -compact, and this is $(n-1)\text{Pr}_A$. $\square$

In particular, for any $n \geq 0$, we get a six-functor formalism

$$\text{Corr}(C) \rightarrow \text{Pr}^L : \text{Spec}(A) \mapsto n\text{Pr}_A$$

for the class E of all morphisms, all of which are regarded as “proper”. If we restrict E to the countably presented maps, this factors as

$$\text{Corr}(C, E) \rightarrow (n+1)\text{Pr} : \text{Spec}(A) \mapsto n\text{Pr}_A.$$

4.2. Ambidexterity. It turns out that for $n \geq 1$, one has not only well-behaved right adjoints, but also well-behaved (and identical!) left adjoints. This phenomenon, of functors having identical left and right adjoints, is known as “ambidexterity”, a term introduced by Hopkins–Lurie [HL13]. Before addressing this, let us quickly recall the notion of adjoints in $(\infty, 2)$ -categories.

DEFINITION 4.2. Let $\mathbb{C}$ be an $(\infty, 2)$ -category.¹⁰ Let $F : X \rightarrow Y$ be a map in $\mathbb{C}$.

The map F is a left adjoint if there is a map $F^R : Y \rightarrow X$ in $\mathbb{C}$ such that there exist maps

$$\eta : \text{id}_X \rightarrow F^R F, \epsilon : F F^R \rightarrow \text{id}_Y$$

in the $(\infty, 1)$ -categories $\text{End}_{\mathbb{C}}(X)$ resp. $\text{End}_{\mathbb{C}}(Y)$, such that the composites

$$F \xrightarrow{F\eta} F F^R F \xrightarrow{\epsilon F} F, F^R \xrightarrow{\eta F^R} F^R F F^R \xrightarrow{F^R \epsilon} F^R$$

are equivalent to the identity.

Dually, F is a right adjoint if there is a map $F^L : Y \rightarrow X$ in $\mathbb{C}$ such that there exist maps

$$\eta : \text{id}_X \rightarrow F F^L, \epsilon : F^L F \rightarrow \text{id}_Y$$

¹⁰The reader is invited to take this to mean $\mathbb{C} \in 2\text{Pr}$ for the present discussion, but the definition makes sense for any reasonable notion of $(\infty, 2)$ -category.

in the $(\infty, 1)$ -categories $\text{End}_{\mathbb{C}}(X)$ resp. $\text{End}_{\mathbb{C}}(Y)$, such that the composites

$$F^L \xrightarrow{F^L \eta} F^L F F^L \xrightarrow{\epsilon F^L} F^L, F \xrightarrow{\eta F} F F^L F \xrightarrow{F \epsilon} F$$

are equivalent to the identity.

If F is a left adjoint, the right adjoint F^R equipped with η and ϵ is unique up to isomorphism, and its left adjoint is given by $(F^R)^L = F$ (with the identical η and ϵ). Indeed, if $((F^R)')^L, \eta', \epsilon'$ is another right adjoint, then the composite

$$F^R \xrightarrow{\eta' F^R} (F^R)' F F^R \xrightarrow{(F^R)'\epsilon} (F^R)'$$

is an isomorphism, with inverse

$$(F^R)' \xrightarrow{\eta(F^R)'} F^R F (F^R)' \xrightarrow{F^R \epsilon} F^R.$$

REMARK 4.3. In our formalism of presentable (∞, n) -categories, there are forgetful functors $n\text{Pr} \rightarrow k\text{Pr}$ for any $k \leq n$. Indeed, there is a symmetric monoidal functor $\text{Ani} \rightarrow 1\text{Pr}$ (as Ani is the unit object) which inductively yields symmetric monoidal functors $n\text{Pr} \rightarrow (n+1)\text{Pr}$ by passing to modules, and hence as right adjoints forgetful functors $(n+1)\text{Pr} \rightarrow n\text{Pr}$. In particular, the preceding definition can be applied to objects of $n\text{Pr}$ for any $n \geq 2$.

We can now state the following general ambidexterity result.¹¹

PROPOSITION 4.4. *Let $\mathbb{C}$ be an $(\infty, 3)$ -category. Let $F : X \rightarrow Y$ be a morphism in $\mathbb{C}$. The following conditions are equivalent.*

- (i) *The map F admits a right adjoint $F^R : Y \rightarrow X$ and both $\eta : \text{id}_X \rightarrow F^R F$ in the $(\infty, 2)$ -category $\text{End}_{\mathbb{C}}(X)$, and $\epsilon : F F^R \rightarrow \text{id}_Y$ in the $(\infty, 2)$ -category $\text{End}_{\mathbb{C}}(Y)$, admit right adjoints.*
- (ii) *The map F admits a left adjoint $F^L : Y \rightarrow X$ and both $\eta : \text{id}_Y \rightarrow F F^L$ in the $(\infty, 2)$ -category $\text{End}_{\mathbb{C}}(Y)$, and $\epsilon : F^L F \rightarrow \text{id}_X$ in the $(\infty, 2)$ -category $\text{End}_{\mathbb{C}}(X)$, admit left adjoints.*

In this case, there is a canonical identification $F^L \cong F^R$.

REMARK 4.5. There is a dual version: We can replace (i) by the condition that the right adjoint F^R of F exists, and that η and ϵ admit left adjoints; one then has to replace (ii) by the condition that the left adjoint F^L of F exists, and that η and ϵ admit right adjoints. Again, this yields a canonical identification $F^L \cong F^R$.

However, one must ask that η and ϵ admit the same type of adjoint for this result to apply.

PROOF. We prove that (i) implies (ii), with $F^L = F^R$; the dual argument proves the converse. To see that F^R is a left adjoint of F , we need to find $\eta' : \text{id} \rightarrow F F^R$ and $\epsilon' : F^R F \rightarrow \text{id}$, satisfying the triangle conditions. But we can simply take $\eta' = \epsilon^R$ and $\epsilon' = \eta^R$. The triangle conditions follow as forming right adjoints is compatible with composition. Moreover, by their definition as right adjoints, η' and ϵ' admit left adjoints, giving (ii). $\square$

¹¹This is certainly very old, but I don't know the original reference. One reference is [Lur09a, Remark 3.4.22].

COROLLARY 4.6. *Let $f : Y = \operatorname{Spec}(B) \rightarrow X = \operatorname{Spec}(A)$ be a map in $\mathcal{C}$. Assume that B is a countably presented A -algebra. For $n \geq 1$, the functor $f_n^* : n\operatorname{Pr}_A \rightarrow n\operatorname{Pr}_B$ admits an $n\operatorname{Pr}_A$ -linear left adjoint $f_{n\sharp} : n\operatorname{Pr}_B \rightarrow n\operatorname{Pr}_A$, commuting with base change. Moreover, there is a canonical identification $f_{n\sharp} \cong f_{n*}$.*

PROOF. We apply the preceding proposition with $\mathbb{C} = (n+1)\operatorname{Pr}_A$ (which lies in $(n+2)\operatorname{Pr}$ and hence in $3\operatorname{Pr}$, as $n \geq 1$), to the functor $f_n^* : n\operatorname{Pr}_A \rightarrow n\operatorname{Pr}_B$. We already know that there is a right adjoint $f_{n*} : n\operatorname{Pr}_B \rightarrow n\operatorname{Pr}_A$ in $(n+1)\operatorname{Pr}_A$. We need to see that

$$\eta : \operatorname{id} \rightarrow f_{n*} f_n^*$$

in $n\operatorname{Pr}_A = \operatorname{End}_{(n+1)\operatorname{Pr}_A}(n\operatorname{Pr}_A, n\operatorname{Pr}_A)$ and

$$\epsilon : f_n^* f_{n*} \rightarrow \operatorname{id}$$

in $\operatorname{End}_{(n+1)\operatorname{Pr}_A}(n\operatorname{Pr}_B, n\operatorname{Pr}_B)$ admit right adjoints.

For η , note that all functors in sight are $n\operatorname{Pr}_A$ -linear, so η is fully determined by its effect on the unit, which is

$$f_{n-1}^* : (n-1)\operatorname{Pr}_A \rightarrow (n-1)\operatorname{Pr}_B$$

in $n\operatorname{Pr}_A$. But as $n \geq 1$, this still has a right adjoint in $n\operatorname{Pr}_A$.

For ϵ , we consider the base change diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\Delta} & Y \times_X Y & \xrightarrow{q} & Y \\ & & p \downarrow & & \downarrow f \\ & & Y & \xrightarrow{f} & X. \end{array}$$

Then we have the map $\eta_\Delta : \operatorname{id} \rightarrow \Delta_{n,*} \Delta_n^*$ in $\operatorname{End}_{(n+1)\operatorname{Pr}_{B \otimes_A B}}(n\operatorname{Pr}_{B \otimes_A B})$, and $p_{n,*} \circ \eta_\Delta \circ q_n^*$ yields a map

$$p_{n,*} \eta_\Delta q_n^* : f_n^* f_{n*} = p_{n,*} q_n^* \rightarrow p_{n,*} \Delta_{n,*} \Delta_n^* q_n^* = \operatorname{id}_{n,*} \operatorname{id}_n^* = \operatorname{id}.$$

This is in fact ϵ : To see this, it suffices to show that η and $p_{n,*} \circ \eta_\Delta \circ q_n^*$ satisfy the triangle identities, which is a consequence of base change (and holds in any six-functor formalism). But η_Δ admits a right adjoint, and hence so does $p_{n,*} \circ \eta_\Delta \circ q_n^*$.

To show that $f_{n\sharp}$ commutes with base change, we apply the same argument in $\operatorname{Fun}(\Delta^1, (n+1)\operatorname{Pr}_A)$ (noting that admitting a left adjoint in $\operatorname{Fun}(\Delta^1, \mathbb{C})$ is equivalent to the assertion that both values admit left adjoints, and these satisfy the Beck–Chevalley base change condition). In fact, applying it in $\operatorname{Fun}(C^{\aleph_1, \operatorname{op}}, (n+1)\operatorname{Pr})$ shows that $f_{n\sharp} \cong f_{n*}$ as functors $C^{\aleph_1, \operatorname{op}} \rightarrow (n+1)\operatorname{Pr}$. $\square$

Thus, for $n \geq 1$, “all maps are étale and proper” (or at least all countably presented maps). A particular consequence is the following dualizability statement (that can also be verified directly).

COROLLARY 4.7. *Let $f : A \rightarrow B$ be a countably presented map of $\mathbb{E}_\infty$ -algebras. For $n \geq 1$, the object $f_{n*}(1) = (n-1)\operatorname{Pr}_B \in n\operatorname{Pr}_A$ is dualizable.*

PROOF. Indeed, proper pushforward along cohomologically smooth maps preserves dualizable objects. Concretely, $(n-1)\operatorname{Pr}_B$ is selfdual, with evaluation

$$(n-1)\operatorname{Pr}_B \otimes_{n\operatorname{Pr}_A} (n-1)\operatorname{Pr}_B = (n-1)\operatorname{Pr}_{B \otimes_A B} \xrightarrow{\Delta_{n-1}^*} (n-1)\operatorname{Pr}_B \xrightarrow{f_{n-1,*}} (n-1)\operatorname{Pr}_B$$

and coevaluation

$$(n-1)\mathrm{Pr}_A \xrightarrow{f_{n-1}^*} (n-1)\mathrm{Pr}_B \xrightarrow{\Delta_{n-1,*}} (n-1)\mathrm{Pr}_{B \otimes_A B} = (n-1)\mathrm{Pr}_B \otimes_{n\mathrm{Pr}_A} (n-1)\mathrm{Pr}_B.$$

□

4.3. Descent. As a final topic of this lecture, we discuss the descent properties of $A \mapsto n\mathrm{Pr}_A$. A standard choice for a Grothendieck topology on C is the fpqc topology, where covers are faithfully flat maps. Here a map $A \rightarrow B$ of $\mathbb{E}_\infty$ -algebras is faithfully flat if $\pi_0 A \rightarrow \pi_0 B$ is a faithfully flat map of static rings, and for all $i \in \mathbb{Z}$ the map $\pi_i A \otimes_{\pi_0 A} \pi_0 B \rightarrow \pi_i B$ is an isomorphism.

The following theorem is essentially due to Mathew [Mat16, Corollary 3.33, Proposition 3.45]. We reproduce some of his arguments below.

THEOREM 4.8. *For any $n \geq 0$, the functor*

$$C^{\mathrm{op}} \rightarrow 1\mathrm{Pr} : \mathrm{Spec}(A) \mapsto n\mathrm{Pr}_A$$

is a hypersheaf in the fpqc topology.

REMARK 4.9. The experts may be confused that there is no assumption here that the fpqc cover is countably presented. The reason is that we replaced Pr^L by $\mathrm{Pr}_{\aleph_1}^L$: It turns out that making this “countability” assumption on the categories, we do not need to put it on the fpqc covers.

We note that in the statement, we could replace the target $1\mathrm{Pr}$ by Pr^L or (large) $(\infty, 1)$ -categories: The sheaf condition only involves countable limits, and those are computed naively. What is important, however, is that already for $n = 1$, we send A only to $\mathrm{Mod}_{D(A)}(\mathrm{Pr}_{\aleph_1}^L)$, not the full $\mathrm{Mod}_{D(A)}(\mathrm{Pr}^L)$: In that case, there are counterexamples due to Zelich [Zel24] and Aoki [Aok24].

PROOF. As $A \mapsto n\mathrm{Pr}_A$ commutes with finite products, it suffices to see that for any fpqc hypercover $A \rightarrow A_\bullet$ of $\mathbb{E}_\infty$ -rings, the map

$$n\mathrm{Pr}_A \rightarrow \lim_{\Delta} n\mathrm{Pr}_{A_\bullet}$$

is an isomorphism (in $1\mathrm{Pr}$, or equivalently in $(n+1)\mathrm{Pr}_A = \mathrm{Mod}_{n\mathrm{Pr}_A}(1\mathrm{Pr})$). First, we reduce to the countably presented case. Note that any faithfully flat map is an $\aleph_1$ -filtered colimit of countably presented faithfully flat maps, cf. Lemma 4.10 below. This formally implies a similar statement for hypercovers. But $1\mathrm{Pr}$ is $\aleph_1$ -compactly generated and hence $\aleph_1$ -filtered colimits commute with countable limits. This lets us pull the $\aleph_1$ -filtered colimit out of the countable limit, and hence reduce to the case that A_i are countably presented A -algebras.

We now start by proving the key non-formal input. In this countably presented hypercover situation, under $!$ -pullback (the right adjoint functors to $*$ -pushforward) the functor

$$D(A) \rightarrow \lim_{\Delta}^! D(A_\bullet)$$

is an equivalence. We first check fully faithfulness. As the displayed functor has a left adjoint, given as the geometric realization of the forgetful functors $D(A_\bullet) \rightarrow D(A)$, we need to see that for all $M \in D(A)$, the map

$$M \rightarrow \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Hom}_A(A_\bullet, M)$$

is an isomorphism. For this, in turn, it suffices to see that the map

$$A \rightarrow \lim_{\Delta} A_\bullet$$

is a pro-isomorphism

$$A \cong \{\lim_{\Delta \leq m} A_\bullet\}_m.$$

Passing to connective covers (which preserve fpqc maps; this implies that if $A \rightarrow A_\bullet$ is an fpqc hypercover, then also $\tau_{\geq 0}A \rightarrow \tau_{\geq 0}A_\bullet$ is an fpqc hypercover, which base changes to $A \rightarrow A_\bullet$), we can assume that A and all A_i are connective. It suffices to see that the cofibre of $A \rightarrow \lim_{\Delta \leq m} A_\bullet$ is a countably presented flat A -module in degree $m-1$; indeed, their transition maps are then ghost maps (as countably presented flat modules are sequential colimits of finite free modules, and maps between finite free modules in different degrees are zero when the source is more coconnective) and a composite of two ghost maps is zero (as ghost maps come from a $\lim^1$, and those would multiply to a $\lim^2$, but that vanishes). These assertions can be checked after base change to $\pi_0 A$, i.e. we can reduce to the static case. But if $A \rightarrow A_\bullet$ is an fpqc hypercover of static rings, then

$$0 \rightarrow A \rightarrow A_0 \rightarrow A_1 \rightarrow \dots$$

is exact, and in particular for each m the cone of $A \rightarrow \lim_{\Delta \leq m} A_\bullet$ agrees with the cokernel of $A_{m-1} \rightarrow A_m$, shifted appropriately. Moreover, the same holds true after any base change in A , in particular along a quotient map $A \rightarrow A/I$; this shows that this cokernel is flat over A .

Thus,

$$D(A) \rightarrow \lim_{\Delta}^! D(A_\bullet)$$

is fully faithful. To prove essential surjectivity, we use a trick. First, it is enough to consider the case of a Čech cover: Indeed, the descent becomes effective after base change along $A \rightarrow A_0$ as then the hypercover is split, so it is enough to do Čech descent along $A \rightarrow A_0$. We need to see that the functor

$$\operatorname{colim}_* D(A_\bullet) \rightarrow D(A)$$

is an isomorphism in $\operatorname{Pr}_{D(A)}^L$ (as this colimit in presentable $(\infty, 1)$ -categories evaluates to the limit along upper-!). After base change along $-\otimes_{D(A)} D(A_0)$, this holds as the cover is split. It follows that it also holds true after tensoring with $\operatorname{colim}_* D(A_\bullet)$. But by fully faithfulness of

$$D(A) \rightarrow \lim_{\Delta}^! D(A_\bullet) = \operatorname{colim}_* D(A_\bullet),$$

we know that the map $\operatorname{colim}_* D(A_\bullet) \rightarrow D(A)$ is a localization, and in particular there is some $M \in \operatorname{colim}_* D(A_\bullet)$ mapping to the unit. This yields a splitting $D(A) \rightarrow \operatorname{colim}_* D(A_\bullet)$, writing $D(A)$ as a retract of $\operatorname{colim}_* D(A_\bullet)$ in $\operatorname{Pr}_{D(A)}^L$. Thus, the map $\operatorname{colim}_* D(A_\bullet) \rightarrow D(A)$ becomes an isomorphism after tensoring over $D(A)$ with the retract $D(A)$ of $\operatorname{colim}_* D(A_\bullet)$. But tensoring with the unit does not do anything, so the map is an isomorphism.

Finally, we can prove that for $n \geq 1$ and any countably presented fpqc hypercover $A \rightarrow A_\bullet$, the functor

$$\operatorname{Pr}_A \rightarrow \lim_{\Delta}^* \operatorname{Pr}_{A_\bullet}$$

is an equivalence, by induction on n . For any $n \geq 1$, the functor $f_{i,n}^* : n\operatorname{Pr}_A \rightarrow n\operatorname{Pr}_{A_i}$ has an $n\operatorname{Pr}_A$ -linear left adjoint $f_{i,n\sharp}$. This formally implies that the functor

$$F_n^* : n\operatorname{Pr}_A \rightarrow \lim_{\Delta} n\operatorname{Pr}_{A_\bullet}$$

has an $n\operatorname{Pr}_A$ -linear left adjoint $F_{n\sharp}$ given by a geometric realization of the left adjoints at the individual levels; see for example [HY17, Corollary 1.3]. To prove fully faithfulness of F_n^* , it suffices to see that the counit map $F_{n\sharp} F_n^* \rightarrow \operatorname{id}$ is an isomorphism. By linearity of all functors, it suffices to see this when evaluated at the unit. If $n = 1$, this is precisely the $!$ -descent for $D(A)$

established above, noting that the colimit defining $F_{1\#}(1)$ is $\text{colim}_* D(A_\bullet)$ computed in presentable $(\infty, 1)$ -categories, which amounts to the limit along upper-!. If $n \geq 2$, then by Corollary 4.7 and Lemma 4.11 below, the value $F_{n\#}(1)$ is dualizable; thus, it suffices to see that the dual map $1 \rightarrow F_{n,*}(1)$ is an isomorphism. But this is precisely the same assertion one categorical level lower, that

$$F_{n-1}^* : (n-1)\text{Pr}_A \rightarrow \lim_{\Delta} (n-1)\text{Pr}_{A_\bullet}$$

is an isomorphism.

This yields fully faithfulness of F_n^* . For essential surjectivity, we can again reduce to a Čech cover given by the map $A \rightarrow A_0$. We need to see that the unit map $\text{id} \rightarrow F_n^* F_{n\#}$ is also an isomorphism. But this holds after base change along $A \rightarrow A_0$, and the pullback functor $n\text{Pr}_{A_0} \rightarrow n\text{Pr}_{A_0 \otimes_A A_0}$ is conservative by the fully faithfulness already established. $\square$

LEMMA 4.10. *Let $f : A \rightarrow B$ be a faithfully flat map of $\mathbb{E}_\infty$ -rings. Then f is an $\aleph_1$ -filtered colimit of countably presented faithfully flat maps $f_i : A \rightarrow B_i$.*

PROOF. First, if A is static, then so is B . This case is due to Waterhouse [Wat75], but let us quickly recall the argument. By Lazard's theorem, flat A -modules are the Ind-category of finite projective A -modules, so we can write the underlying module of B as an $\aleph_1$ -filtered colimit of countably presented flat modules M_i . The multiplication map $B \otimes B \rightarrow B$ gives maps $M_i \otimes M_i \rightarrow M_j$ for some $j = j(i)$, and inductively choosing a sequence $i_0, i_1, \dots$ so that $M_{i_n} \otimes M_{i_n} \rightarrow M_{i_{n+1}}$ is well-defined, the colimit $\text{colim}(M_{i_0} \rightarrow M_{i_1} \rightarrow \dots)$ admits a multiplication, that will also be commutative and associative and unital up to passing to a cofinal system. This writes B in the desired form.

In general, write f as the $\aleph_1$ -filtered colimit of all countably presented maps A -algebras B_i mapping to B . We want to see that the set of i for which $A \rightarrow B_i$ is faithfully flat is cofinal. We get a similar presentation of $\pi_0 A$ as the colimit of the $\pi_0 B_i$, and by the static case, for any i there is some j so that $\pi_0 B_i \rightarrow \pi_0 B_j$ factors over a faithfully flat $\pi_0 A$ -algebra. Taking a sequence $i_0, i_1, \dots$ so that each map $\pi_0 B_{i_0} \rightarrow \pi_0 B_{i_1} \rightarrow \dots$ factors over a faithfully flat $\pi_0 A$ -algebra, the colimit $B_j = \text{colim}(B_{i_0} \rightarrow B_{i_1} \rightarrow \dots)$ is still countably presented, and has the property that $\pi_0 A \rightarrow \pi_0 B_j$ is faithfully flat. Using a similar argument, we can arrange that $\pi_0 B_i \otimes_{\pi_0 A} \pi_n A \rightarrow \pi_n B_i$ is an isomorphism for any given i , and hence all i by a further countable colimit. $\square$

LEMMA 4.11. *Let $\mathcal{V} \in \text{CAlg}(1\text{Pr})$. Consider a countable diagram $C : I \rightarrow \text{Mod}_{\mathcal{V}}(1\text{Pr})$ such that for all $i \in I$, the $\mathcal{V}$ -linear category $C(i) \in \text{Mod}_{\mathcal{V}}(1\text{Pr})$ is dualizable, and for all maps $i \rightarrow j$ in I , the functor $C(i) \rightarrow C(j)$ is a left adjoint in $\text{Mod}_{\mathcal{V}}(1\text{Pr})$. Then $\text{colim}_I C \in \text{Mod}_{\mathcal{V}}(1\text{Pr})$ is dualizable, with dual $\text{colim}_I C^\vee$ where $C^\vee : I \rightarrow \text{Mod}_{\mathcal{V}}(1\text{Pr})$ is the functor obtained from C by passing to right adjoints to get a functor $C^{\text{op}} \rightarrow \text{Mod}_{\mathcal{V}}(1\text{Pr})$, and then dualizing.*

PROOF. See [Ram24, Proposition 1.62], or the appendix to this lecture. We note that in the proof above, we only really need that $\text{colim}_I C$ is reflexive (i.e., it maps isomorphically to its bidual). Its dual is given by $\lim_{I^{\text{op}}} C^\vee$. But all transition functors here are right adjoints, so by Theorem 3.4, this agrees with $\text{colim}_I C^\vee$ where the transition functors are the left adjoints. One can then apply the same reasoning again, showing that the bidual is given by $\text{colim}_I C$. It remains to check that the induced map $\text{colim}_I C \rightarrow \text{colim}_I C$ is the identity; this is best seen by writing down the pairing between $\text{colim}_I C$ and $\text{colim}_I C^\vee$ that induces these identifications. $\square$

Appendix to Lecture IV: Dualizable Categories

In this appendix, we collect various results on dualizable categories. These have recently been used to great effect by Efimov [Efi25]. Ramzi's paper [Ram24] is an excellent reference for these results.

Let us start by recalling the “standard” setting of dualizable presentable stable $(\infty, 1)$ -categories. We will then explain the modifications to pass from the stable (equivalently, $D(\mathbb{S})$ -linear) case to $\mathcal{V}$ -linear presentable $(\infty, 1)$ -categories for any $\mathcal{V} \in \mathbf{CAlg}(1\mathbf{Pr})$; and from the full $\mathbf{Pr}^L$ to its fragment $1\mathbf{Pr} = \mathbf{Pr}_{\aleph_1}^L$.

We start with the following important example.

PROPOSITION 4.12 (Lurie). *Let $\mathcal{C}$ be a compactly generated presentable stable $(\infty, 1)$ -category. Then $\mathcal{C}$ is dualizable in $\mathbf{Pr}_{\mathrm{st}}^L$, and the dual is given by*

$$\mathcal{C}^\vee = \mathrm{Ind}(\mathcal{C}_0^{\omega, \mathrm{op}}),$$

with the pairing

$$\mathcal{C} \otimes \mathcal{C}^\vee \rightarrow D(\mathbb{S})$$

uniquely determined by its effect on compact objects, which is $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}^{D(\mathbb{S})}(Y, X)$.

PROOF. We need to see that for any other $\mathcal{D} \in \mathbf{Pr}_{\mathrm{st}}^L$, the natural functor

$$\mathcal{C}^\vee \otimes \mathcal{D} \rightarrow \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

is an equivalence. But $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega)$, so $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is given by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^\omega, \mathcal{D})$. By the formula for the Lurie tensor product, this is also

$$\mathrm{Ind}(\mathcal{C}_0^{\omega, \mathrm{op}}) \otimes \mathcal{D}.$$

□

Being dualizable is a bit more general:

THEOREM 4.13 (Lurie, [Lur18b, Proposition D.7.3.1]). *Let $\mathcal{C}$ be a presentable stable $(\infty, 1)$ -category. The following conditions are equivalent.*

- (i) *The object $\mathcal{C} \in \mathbf{Pr}_{\mathrm{st}}^L$ is dualizable.*
- (ii) *Inside $\mathbf{Pr}_{\mathrm{st}}^L$, $\mathcal{C}$ is a retract of a compactly generated presentable stable $(\infty, 1)$ -category.*
- (iii) *The $(\infty, 1)$ -category $\mathcal{C}$ is $\aleph_1$ -presentable, and the functor*

$$\mathrm{Ind}(\mathcal{C}^{\aleph_1}) \rightarrow \mathcal{C}$$

has a left adjoint.

- (iv) *The presentable $(\infty, 1)$ -category $\mathcal{C}$ is compactly assembled, cf. Definition 4.14 below.*

Let us recall the notion of being compactly assembled.

DEFINITION 4.14. Let $\mathcal{C}$ be a presentable $(\infty, 1)$ -category.

- (i) A morphism $f : Y \rightarrow X$ in $\mathcal{C}$ is compact if for any filtered colimit $Z = \mathrm{colim}_i Z_i$, there is an oblique arrow making the following diagram commute:

$$\begin{array}{ccc} \mathrm{colim}_i \mathrm{Hom}(X, Z_i) & \longrightarrow & \mathrm{Hom}(X, Z) \\ \downarrow & \swarrow \text{---} & \downarrow \\ \mathrm{colim}_i \mathrm{Hom}(Y, Z_i) & \longrightarrow & \mathrm{Hom}(Y, Z) \end{array}$$

- (ii) The presentable $(\infty, 1)$ -category is compactly assembled if it is generated under colimits by sequential colimits of the form

$$\operatorname{colim}(X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots)$$

where all maps $X_n \rightarrow X_{n+1}$ are compact.

Intuitively, a compact morphism is a morphism factoring over a compact object, but there may not exist, in $\mathcal{C}$, such a compact object. Still, sequential colimits $X = \operatorname{colim}(X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots)$ along compact maps are $\aleph_1$ -compact, and hence compactly assembled $\mathcal{C}$ are $\aleph_1$ -presentable. Indeed, for any $\aleph_1$ -filtered colimit $Z = \operatorname{colim}_i Z_i$, one has

$$\begin{aligned} \operatorname{Hom}(X, Z) &= \lim_n \operatorname{Hom}(X_n, Z) = \lim_n (\operatorname{colim}_i \operatorname{Hom}(X_n, Z_i)) \\ &= \operatorname{colim}_i (\lim_n \operatorname{Hom}(X_n, Z_i)) = \operatorname{colim}_i \operatorname{Hom}(X, Z_i), \end{aligned}$$

using in the second equality that the two pro-systems $\operatorname{Hom}(X_n, Z)$ and $\operatorname{colim}_i \operatorname{Hom}(X_n, Z)$ are pro-isomorphic, by definition of compact maps, and in the third equality that countable limits commute with $\aleph_1$ -filtered colimits.

EXAMPLE 4.15. Let $\mathcal{C}$ be the category (in fact, partially ordered set) of open subsets of $\mathbb{R}$. Then a map $U \subset V$ in $\mathcal{C}$ is compact if and only if it is a strict inclusion $U \Subset V$, i.e. the closure of U in V is compact, i.e. it factors over a compact subset of $\mathbb{R}$. Then $\mathcal{C}$ is compactly assembled, as any open subset U of $\mathbb{R}$ is a countable union of subsets $U_n \subset U$ along strict inclusions $U_n \Subset U_{n+1}$.

Passing to sheaves, it follows that also $\operatorname{Shv}(\mathbb{R}, D(\mathbb{S}))$ is compactly assembled and hence dualizable (but not compactly generated).

EXAMPLE 4.16. In condensed mathematics, there are various examples of such compactly assembled $\mathcal{C}$ of functional-analytic nature, for example nuclear $\mathbb{Z}_p$ -modules, for which compact morphisms are precisely the compact morphisms known from functional analysis.

As preparation for the proof of Theorem 4.13, we note the following.

LEMMA 4.17. *Let $\mathcal{C}$ be a κ -presentable $(\infty, 1)$ -category. Then $\mathcal{C}$ is compactly assembled if and only if the functor*

$$\operatorname{Ind}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$$

has a left adjoint.

Moreover, this property passes to retracts in Pr^L .

PROOF. In the forward direction, assume that $\mathcal{C}$ is compactly assembled. Then the functor

$$\operatorname{Ind}(\mathcal{C}^{\aleph_1}) \rightarrow \mathcal{C}$$

has a fully faithful left adjoint, taking a sequential colimit $X = \operatorname{colim}_n X_n$ along compact transition maps to “ colim ” $_n X_n$. Indeed, for any filtered colimit $Z = \operatorname{colim}_i Z_i$, we have

$$\operatorname{Hom}(X, Z) = \lim_n \operatorname{Hom}(X_n, Z) = \lim_n (\operatorname{colim}_i \operatorname{Hom}(X_n, Z_i)) = \operatorname{Hom}(\text{“colim”}_n X_n, \text{“colim”}_i Z_i),$$

using in the second equality that $(\operatorname{Hom}(X_n, Z))_n$ and $(\operatorname{colim}_i \operatorname{Hom}(X_n, Z_i))_n$ are pro-isomorphic. Via composition with the embedding $\operatorname{Ind}(\mathcal{C}^{\aleph_1}) \rightarrow \operatorname{Ind}(\mathcal{C}^\kappa)$, this gives the left adjoint in general.

Conversely, assume that the left adjoint exists. Take any $X \in \mathcal{C}$, with left adjoint “ colim ” $_i X_i$. Then for any i , the map $X_i \rightarrow X$ is compact. Indeed, for any filtered colimit $Z = \operatorname{colim}_j Z_j$, one has

$$\operatorname{Hom}(X, Z) = \operatorname{Hom}(\text{“colim”}_i X_i, \text{“colim”}_j Z_j)$$

and hence a map to $\text{Hom}(X_i, \text{"colim"}_j Z_j) = \text{colim}_j \text{Hom}(X_i, Z_j)$, making the desired diagram commute. This yields sufficiently many compact maps to ensure that $\mathcal{C}$ is compactly assembled.

In any 2-category with idempotent-complete morphism categories, a retract (inside the category of arrows in the 2-category) of a morphism admitting a left adjoint also admits a left adjoint (cf. [Ram24, Lemma 1.47]), showing stability of this condition under retracts. $\square$

PROOF OF THEOREM 4.13. As dualizability is stable under retracts, Proposition 4.12 shows that (ii) implies (i). It is also clear that (iii) implies (ii) (as the left adjoint is necessarily fully faithful, as its right adjoint is the Yoneda embedding, which is fully faithful). Moreover, (iv) implies (iii). Indeed, compactly assembled implies $\aleph_1$ -presentable, and to show that the left adjoint to $\text{Ind}(\mathcal{C}^{\omega_1}) \rightarrow \mathcal{C}$ exists, it suffices to construct it on the generators $X = \text{colim}(X_0 \rightarrow X_1 \rightarrow \dots)$ (with compact transition maps). We can assume that all X_i are $\aleph_1$ -compact, as any compact map factors over an $\aleph_1$ -compact object in an $\aleph_1$ -compactly generated $(\infty, 1)$ -category. Then we claim that the left adjoint sends X to the formal colimit $\text{"colim"}(X_0 \rightarrow X_1 \rightarrow \dots) \in \text{Ind}(\mathcal{C}^{\aleph_1})$. Indeed, for any filtered colimit $Z = \text{colim}_i Z_i$, we have

$$\text{Hom}(X, Z) = \lim_n \text{Hom}(X_n, Z) = \lim_n (\text{colim}_i \text{Hom}(X_n, Z_i)) = \text{Hom}(\text{"colim"}_n X_n, \text{"colim"}_i Z_i),$$

using in the second equality that $(\text{Hom}(X_n, Z))_n$ and $(\text{colim}_i \text{Hom}(X_n, Z_i))_n$ are pro-isomorphic.

It remains to see that (i) implies (iv); the proof of this will loosely reverse the logic of the other steps. Assume that $\mathcal{C}$ is dualizable, and assume it is κ -presentable for some κ . We get the functor

$$\mathcal{D} = \text{Ind}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$$

which is colimit-preserving and has a fully faithful right adjoint. The functor $\mathcal{D} \rightarrow \mathcal{C}$ is a localization in Pr_{st}^L . As localizations are preserved under tensor products, it follows that so is $\mathcal{D} \otimes \mathcal{C}^\vee \rightarrow \mathcal{C} \otimes \mathcal{C}^\vee$ where $\mathcal{C}^\vee$ is the dual of $\mathcal{C}$. But by dualizability of $\mathcal{C}$, this agrees with

$$\text{Fun}^L(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}^L(\mathcal{C}, \mathcal{C}).$$

In particular, the identity functor $\mathcal{C} \rightarrow \mathcal{C}$ lifts to a colimit-preserving functor $\mathcal{C} \rightarrow \mathcal{D}$, thus making $\mathcal{C}$ a retract of the compactly generated $\mathcal{D}$. The rest follows from Lemma 4.17. $\square$

Let us now first note the modifications required to deal with dualizable objects of Pr_κ^L , for any κ . More precisely, we have the following.

DEFINITION 4.18. Let $(\text{Pr}_{\text{st}}^L)^{\text{dual}}$ be the non-full subcategory consisting of dualizable presentable stable $(\infty, 1)$ -categories, and colimit-preserving functors whose right adjoint is still colimit-preserving.

THEOREM 4.19 (Efimov, Ramzi). *The $(\infty, 1)$ -category $(\text{Pr}_{\text{st}}^L)^{\text{dual}}$ admits all small colimits (and they commute with the forgetful functor to Pr_{st}^L), and is $\aleph_1$ -presentable. For any uncountable regular cardinal κ , the following conditions on $\mathcal{C} \in (\text{Pr}_{\text{st}}^L)^{\text{dual}}$ are equivalent:*

- (i) *It is a κ -compact object in $(\text{Pr}_{\text{st}}^L)^{\text{dual}}$.*
- (ii) *It is a κ -compact object in $\text{Pr}_{\kappa, \text{st}}^L$.*
- (iii) *It is dualizable in $\text{Pr}_{\kappa, \text{st}}^L$.*

In particular, all dualizable presentable stable $(\infty, 1)$ -categories $\mathcal{C}$ are objects of $\text{Pr}_{\aleph_1, \text{st}}^L$, and they are dualizable in there if and only if they are $\aleph_1$ -compact as objects of $(\text{Pr}_{\text{st}}^L)^{\text{dual}}$. Thus,

$$(\text{Pr}_{\text{st}}^L)^{\text{dual}} = \text{Ind}_{\aleph_1}(\text{Pr}_{\aleph_1, \text{st}}^L)^{\text{dual}}.$$

In particular, we see that dualizability in $\mathrm{Pr}_{\aleph_1, \mathrm{st}}^L$ also includes a weak finiteness condition (namely countable presentation).

PROOF. Any dualizable $\mathcal{C}$ is the retract $\mathcal{C} \rightarrow \mathrm{Ind}(\mathcal{C}^{\aleph_1}) \rightarrow \mathcal{C}$, and the formation of this retract is functorial on $(\mathrm{Pr}_{\mathrm{st}}^L)^{\mathrm{dual}}$. (To check functoriality of $\mathcal{C} \rightarrow \mathrm{Ind}(\mathcal{C}^{\aleph_1})$, it suffices to do so after composing with the embedding $\mathrm{Ind}(\mathcal{C}^\kappa)$ for any κ . Then it follows as the diagram of right adjoints commutes, choosing κ large enough so that the colimit-preserving right adjoint functor also preserves κ -compact objects.) This already shows that $(\mathrm{Pr}_{\mathrm{st}}^L)^{\mathrm{dual}}$ admits all small colimits (and they are computed in $\mathrm{Pr}_{\mathrm{st}}^L$), as compactly generated $(\infty, 1)$ -categories are stable under small colimits along functors that preserve compact objects.

In fact, this argument writes $(\mathrm{Pr}_{\mathrm{st}}^L)^{\mathrm{dual}}$ as a full subcategory of the $(\infty, 1)$ -category $\mathcal{E}$ of small stable $(\infty, 1)$ -categories $\mathcal{D}$ admitting all countable colimits, equipped with an idempotent endofunctor of $\mathrm{Ind}_{\aleph_1}(\mathcal{D})$ that preserves countable colimits (by sending $\mathcal{C}$ to $\mathcal{D} = \mathcal{C}^{\aleph_1}$ and the restricted left adjoint $\mathcal{C}^{\aleph_1} \rightarrow \mathrm{Ind}_{\aleph_1}(\mathcal{C}^{\aleph_1})$, extended to $\mathrm{Ind}_{\aleph_1}(\mathcal{C}^{\aleph_1})$). The full subcategory is singled out by the condition that the natural map from $\mathcal{D}$ to the retract of $\mathrm{Ind}_{\aleph_1}(\mathcal{D})$ (defined by the idempotent endofunctor) is an equivalence.

Roughly arguing as in Lecture II (cf. also [Ram24, Proposition 3.10]), one sees that $\mathcal{E}$ is $\aleph_1$ -presentable, with $\aleph_1$ -compact objects being those where $\mathcal{D}$ is countably presented. Asking that the map from $\mathcal{D}$ towards this retract is an equivalence amounts to a fibre product of $\aleph_1$ -presentable $(\infty, 1)$ -categories and hence is $\aleph_1$ -presentable; again the $\aleph_1$ -compact objects are those for which $\mathcal{C}^{\aleph_1}$ is countably presented as a small stable $(\infty, 1)$ -category. This is equivalent to $\mathcal{C}$ being $\aleph_1$ -compact as an object of $\mathrm{Pr}_{\aleph_1, \mathrm{st}}^L$. In this case, $\mathcal{C}$ is dualizable in $\mathrm{Pr}_{\aleph_1, \mathrm{st}}^L$, as for $\aleph_1$ -compact $\mathcal{D}$, the internal Hom from $\mathcal{D}$ in $\mathrm{Pr}_{\aleph_1}^L$ agrees with the internal Hom from $\mathcal{D}$ in Pr^L , cf. Proposition 3.8.

Conversely, if $\mathcal{C}$ is dualizable in $\mathrm{Pr}_{\aleph_1, \mathrm{st}}^L$, then it must be $\aleph_1$ -compact, as the unit of $\mathrm{Pr}_{\aleph_1, \mathrm{st}}^L$ is $\aleph_1$ -compact. For general κ , the arguments are the same. $\square$

Now we want to generalize everything to $\mathcal{V}$ -linear presentable $(\infty, 1)$ -categories, for any $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}_{\aleph_1}^L)$.¹²

We work with $\mathrm{Pr}_{\mathcal{V}}^L = \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$ at first, and later discuss the modifications required for $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L = \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}_{\aleph_1}^L)$. A very important notion in the $\mathcal{V}$ -linear case becomes that of admitting a (right) adjoint internally in $\mathrm{Pr}_{\mathcal{V}}^L$: This is equivalent to the right adjoint preserving small colimits, and tensor products with objects of $\mathcal{V}$. Applied to maps out of an object, this yields the notion of $\mathcal{V}$ -atomic objects.

DEFINITION 4.20. Let $\mathcal{C} \in \mathrm{Pr}_{\mathcal{V}}^L$. An object $X \in \mathcal{C}$ is $\mathcal{V}$ -atomic if $\mathrm{Hom}_{\mathcal{C}}^{\mathcal{V}}(X, -) : \mathcal{C} \rightarrow \mathcal{V}$ preserves small colimits, and tensor products with $\mathcal{V}$.

Here and in the following, we use $\mathrm{Hom}^{\mathcal{V}}$ to denote the $\mathcal{V}$ -internal Hom.

We note that as $1 \in \mathcal{V}$ is $\aleph_1$ -compact, all $\mathcal{V}$ -atomic objects are in particular $\aleph_1$ -compact, and hence there is at most a set worth of isomorphism classes of $\mathcal{V}$ -atomic objects.

DEFINITION 4.21. Let $\mathcal{C} \in \mathrm{Pr}_{\mathcal{V}}^L$. Then $\mathcal{C}$ is $\mathcal{V}$ -atomically generated if it is generated under small colimits and tensor products by $\mathcal{V}$ from its $\mathcal{V}$ -atomic objects.

¹²With little change, everything works for $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}_{\kappa}^L)$ for any uncountable regular κ ; this generality is considered in the work of Ramzi [Ram24].

By the above remark, any $\mathcal{V}$ -atomically generated $\mathcal{C}$ is in particular $\aleph_1$ -compactly generated.

By work of, among others, Lurie [Lur17], Gepner–Haugseeng [GH15], Heine [Hei23] and Hinich [Hin23], there is also a notion of $\mathcal{V}$ -enriched $(\infty, 1)$ -category. For any $\mathcal{C} \rightarrow \mathrm{Pr}_{\mathcal{V}}^L$ and any anima S mapping to $\mathcal{C}$, one can endow S with the structure of a $\mathcal{V}$ -enriched $(\infty, 1)$ -category (using the $\mathcal{V}$ -internal Hom's $\mathrm{Hom}_{\mathcal{C}}^{\mathcal{V}}(-, -)$ restricted to S). Conversely, if S is a $\mathcal{V}$ -enriched $(\infty, 1)$ -category, one can build the free $\mathcal{V}$ -linear presentable $(\infty, 1)$ -category generated by S ,

$$\mathcal{P}_{\mathcal{V}}(S) = \mathrm{Fun}_{\mathcal{V}}(S^{\mathrm{op}}, \mathcal{V}).$$

By universality, there is a natural comparison functor

$$\mathcal{P}_{\mathcal{V}}(S) \rightarrow \mathcal{C}$$

when executing these two constructions after another. If S consists of $\mathcal{V}$ -atomic objects, this functor is fully faithful, and it is an equivalence when S generates $\mathcal{C}$ under colimits and tensoring by objects of $\mathcal{V}$. Thus, any $\mathcal{V}$ -atomically generated $\mathcal{C}$ can be written as an enriched presheaf category.

By Reutter–Zetto [RZ25], one can in fact define the notion of enriched $(\infty, 1)$ -category this way, as consisting of $\mathcal{C} \in \mathrm{Pr}_{\mathcal{V}}^L$ equipped with an anima S of $\mathcal{V}$ -atomic objects that generates $\mathcal{C}$ under small colimits and tensoring by $\mathcal{V}$.¹³ Taking this as the definition of enriched $(\infty, 1)$ -category, we can justify the above as follows:

THEOREM 4.22. *Let $\mathcal{C} \in \mathrm{Pr}_{\mathcal{V}}^L$ and S be any anima mapping to $\mathcal{C}$. There is a final $\mathcal{C}_S \in \mathrm{Pr}_{\mathcal{V}}^L$ equipped with a functor $F : \mathcal{C}_S \rightarrow \mathcal{C}$ and a lift $i : S \rightarrow \mathcal{C}_S$ such that the image of $i : S \rightarrow \mathcal{C}_S$ is a collection of $\mathcal{V}$ -atomic generators of $\mathcal{C}_S$. Moreover, for all $X, Y \in S$ the natural map*

$$\mathrm{Hom}_{\mathcal{C}_S}^{\mathcal{V}}(i(X), i(Y)) \rightarrow \mathrm{Hom}_{\mathcal{C}}^{\mathcal{V}}(F(i(X)), F(i(Y)))$$

is an isomorphism in $\mathcal{V}$.

If S consists of $\mathcal{V}$ -atomic objects, the comparison functor $F : \mathcal{C}_S \rightarrow \mathcal{C}$ is fully faithful.

Thus $(\mathcal{C}_S, S)$ is a $\mathcal{V}$ -enriched $(\infty, 1)$ -category in the sense above, and we denote $\mathcal{P}_{\mathcal{V}}(S) := \mathcal{C}_S$.

SKETCH. We can assume that S is a set (by taking any set surjecting onto the given anima, and using that F is fully faithful on S to extend the map back to the anima). We can reduce to the case that S is finite, by noting that filtered colimits of atomically generated categories, along fully faithful functors preserving atoms, are still atomically generated and contain the original categories fully faithfully. If S has just one element, giving an object $X \in \mathcal{C}$, then we get the algebra $\mathcal{A} = \mathrm{End}_{\mathcal{C}}^{\mathcal{V}}(X) \in \mathrm{Alg}(\mathcal{V})$, and $\mathcal{C}_S$ is $\mathrm{Mod}_{\mathcal{A}}(\mathcal{V})$, which is atomically generated by the marked object. For general finite sets S , the argument is similar, except that one has to make sense of “ $\mathcal{V}$ -algebras on many objects”.

Another approach is to note that $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{V}) = \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani}) \otimes_{\mathrm{Ani}} \mathcal{V}$ is the free $\mathcal{V}$ -linear presentable $(\infty, 1)$ -category on S , and hence i extends to a map $i : \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathcal{C}_S$ still denoted i . Then the image of S consists of $\mathcal{V}$ -atomic objects if and only if the right adjoint i^R is still $\mathcal{V}$ -linear, in which case the datum of $S \rightarrow \mathcal{C}_S$ is equivalent (by Barr–Beck) to an algebra in $\mathrm{End}_{\mathcal{V}}(\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{V})) = \mathrm{Fun}(S \times S^{\mathrm{op}}, \mathcal{V})$, which then comes equipped with a natural “convolution/composition” monoidal structure. (In fact, this notion is precisely that of a “ $\mathcal{V}$ -algebra on many objects”, or a $\mathcal{V}$ -enriched $(\infty, 1)$ -category.) On the other hand, the given $F : S \rightarrow \mathcal{C}$ also induces $F : \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathcal{C}$ and has

¹³There are two slightly different notions of enriched $(\infty, 1)$ -categories, depending on whether one asks that S injects into the anima of $\mathcal{V}$ -atomic objects (this is called “univalence”), or is any map. We will allow any map, i.e. not enforce univalence.

a lax $\mathcal{V}$ -linear right adjoint G . Then the composite GF still induces an algebra in $\text{Fun}(S \times S^{\text{op}}, \mathcal{V})$ (as the best $\mathcal{V}$ -linear approximation), which translates to the desired data. $\square$

THEOREM 4.23 (Ramzi, [Ram24, Theorem 1.49, Theorem 3.4]). *Let $\mathcal{C} \in \text{Pr}_{\mathcal{V}}^L$. The following conditions are equivalent.*

- (i) *The $\mathcal{V}$ -linear presentable $(\infty, 1)$ -category $\mathcal{C}$ is dualizable.*
- (ii) *The $\mathcal{V}$ -linear presentable $(\infty, 1)$ -category $\mathcal{C}$ is a retract, in $\text{Pr}_{\mathcal{V}}^L$, of a $\mathcal{V}$ -atomically generated $\mathcal{D}$.*
- (iii) *The presentable $(\infty, 1)$ -category $\mathcal{C}$ is $\aleph_1$ -compactly generated, and the functor*

$$\mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\aleph_1}) \rightarrow \mathcal{C}$$

admits a $\mathcal{V}$ -linear left adjoint.

Unfortunately, there is no general analogue of the condition of being “compactly assembled” in this case. (One can still define such a notion, and it implies being dualizable, but the converse is likely not true.)

PROOF. First, one shows that atomic generation implies dualizability. This is due to Berman [Ber20]; a clean argument in the presentable language is [Ram24, Propositions 1.40, 1.41]. With this, (iii) implies (ii) implies (i) is clear. For the converse, one first shows that (i) implies (ii). In fact, this follows from the same argument as before, that the localization $\mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\kappa}) \rightarrow \mathcal{C}$ is preserved under tensoring with $\mathcal{C}^{\vee}$, and taking a preimage of the identity $\text{id}_{\mathcal{C}} \in \text{Fun}_{\mathcal{V}}^L(\mathcal{C}, \mathcal{C}) \cong \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^{\vee}$. In the atomically generated case, it is clear that (iii) holds. Now we claim that any retract (in Pr^L) of an $\aleph_1$ -presentable $(\infty, 1)$ -category is still $\aleph_1$ -presentable. This is analogous to Lemma 4.17: One shows a priori $\aleph_1$ -compactly assembled, but now a sequential colimit of $\aleph_1$ -compact maps yields an $\aleph_1$ -compact object, so $\aleph_1$ -compactly assembled implies $\aleph_1$ -compactly generated. See also [Ram24, Section 2.3]. Once we know $\aleph_1$ -compact generation, the existence of the left adjoint in (iii) is stable under retracts (cf. [Ram24, Lemma 1.47]), so dualizability implies (iii) in general. $\square$

As before, one can define $(\text{Pr}_{\mathcal{V}}^L)^{\text{dual}}$ as the nonfull subcategory of $\text{Pr}_{\mathcal{V}}^L$ whose objects are dualizable $\mathcal{V}$ -linear presentable $(\infty, 1)$ -categories, and morphisms are those functors that admit right adjoints in $\text{Pr}_{\mathcal{V}}^L$. On this $(\infty, 1)$ -category, the formation of the left adjoint

$$\mathcal{C} \rightarrow \mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\aleph_1})$$

is functorial (it suffices to see this after composing to $\mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\kappa})$ for any κ , and then it follows by looking at the corresponding right adjoints, taking κ large enough so that the right adjoint to $\mathcal{C} \rightarrow \mathcal{D}$ preserves κ -compact objects).

THEOREM 4.24 (Ramzi, [Ram24, Theorem 3.7]). ¹⁴ *The $(\infty, 1)$ -category $(\text{Pr}_{\mathcal{V}}^L)^{\text{dual}}$ admits all small colimits (and they commute with the forgetful functor to $\text{Pr}_{\mathcal{V}}^L$), and is $\aleph_1$ -presentable. For any uncountable regular cardinal κ , the following conditions on $\mathcal{C} \in (\text{Pr}_{\mathcal{V}}^L)^{\text{dual}}$ are equivalent:*

- (i) *It is a κ -compact object in $(\text{Pr}_{\mathcal{V}}^L)^{\text{dual}}$.*
- (ii) *It is a κ -compact object in $\text{Pr}_{\kappa, \mathcal{V}}^L$.*
- (iii) *It is dualizable in $\text{Pr}_{\kappa, \mathcal{V}}^L$.*

¹⁴I believe the characterization of the κ -compact objects in $(\text{Pr}_{\mathcal{V}}^L)^{\text{dual}}$ as being those that are dualizable in $\text{Pr}_{\kappa, \mathcal{V}}^L$ is not stated explicitly in [Ram24]; this addendum basically comes from Pr_{κ}^L being κ -presentable.

In particular, all dualizable presentable $\mathcal{V}$ -linear $(\infty, 1)$ -categories $\mathcal{C}$ are objects of $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$, and they are dualizable in there if and only if they are $\aleph_1$ -compact as objects of $(\mathrm{Pr}_{\mathcal{V}}^L)^{\mathrm{dual}}$. Thus,

$$(\mathrm{Pr}_{\mathcal{V}}^L)^{\mathrm{dual}} = \mathrm{Ind}_{\aleph_1}(\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L)^{\mathrm{dual}}.$$

PROOF. The proof is very similar to the stable case. As observed above, any dualizable $\mathcal{C}$ is the retract $\mathcal{C} \rightarrow \mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\aleph_1}) \rightarrow \mathcal{C}$, and the formation of this retract is functorial on $(\mathrm{Pr}_{\mathcal{V}}^L)^{\mathrm{dual}}$. This already shows that $(\mathrm{Pr}_{\mathcal{V}}^L)^{\mathrm{dual}}$ admits all small colimits (and they are computed in $\mathrm{Pr}_{\mathcal{V}}^L$), as atomatically generated $\mathcal{V}$ -linear $(\infty, 1)$ -categories are stable under small colimits along functors that preserve atomic objects.

In fact, this argument writes $(\mathrm{Pr}_{\mathcal{V}}^L)^{\mathrm{dual}}$ as a full subcategory of the $(\infty, 1)$ -category $\mathcal{E}$ of small $(\infty, 1)$ -categories $\mathcal{D} \in \mathrm{Mod}_{\mathcal{V}^{\aleph_1}}(\mathrm{Rex}_{\aleph_1})$, equipped with an idempotent endofunctor of $\mathcal{P}_{\mathcal{V}}(\mathcal{D})^{\aleph_1}$ in $\mathrm{Mod}_{\mathcal{V}^{\aleph_1}}(\mathrm{Rex}_{\aleph_1})$ (by sending $\mathcal{C}$ to $\mathcal{D} = \mathcal{C}^{\aleph_1}$ and the restricted left adjoint $\mathcal{C}^{\aleph_1} \rightarrow \mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\aleph_1})^{\aleph_1}$, extended to $\mathcal{P}_{\mathcal{V}}(\mathcal{C}^{\aleph_1})^{\aleph_1}$). The full subcategory is singled out by the condition that the natural map from $\mathcal{D}$ to the retract of $\mathcal{P}_{\mathcal{V}}(\mathcal{D})^{\aleph_1}$ (defined by the idempotent endofunctor) is an equivalence.

From this description, one sees that $\mathcal{E}$ is $\aleph_1$ -presentable, with $\aleph_1$ -compact objects being those where $\mathcal{D}$ is countably presented. Asking that the map from $\mathcal{D}$ towards this retract is an equivalence amounts to a fibre product of $\aleph_1$ -presentable $(\infty, 1)$ -categories and hence is $\aleph_1$ -presentable; again the $\aleph_1$ -compact objects are those for which $\mathcal{C}^{\aleph_1}$ is countably presented as an object of $\mathrm{Mod}_{\mathcal{V}^{\aleph_1}}(\mathrm{Rex}_{\aleph_1})$. This is equivalent to $\mathcal{C}$ being $\aleph_1$ -compact as an object of $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$. In this case, $\mathcal{C}$ is dualizable in $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$, as the internal Hom from $\mathcal{C}$ is the same in $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$ and all of $\mathrm{Pr}_{\mathcal{V}}^L$, cf. Proposition 3.8 (which also holds in the $\mathcal{V}$ -linear case).

Conversely, if $\mathcal{C}$ is dualizable in $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$, then it must be $\aleph_1$ -compact, as the unit of $\mathrm{Pr}_{\aleph_1, \mathcal{V}}^L$ is $\aleph_1$ -compact. For general κ , the arguments are the same. $\square$

5. Lecture V: Quasicoherent Sheaves of Higher Categories, II

In this lecture, we extend the previous discussion from affine schemes to stacks.

Let C as before the opposite of $\mathrm{CAlg}(D(\mathbb{S}))$, thought of as affine derived schemes; and let $\tilde{C} = \mathrm{Shv}_{\mathrm{fpqc}}(C)$ be the ∞ -topos of fpqc sheaves on C .¹⁵ By Theorem 4.8, we get for any $n \geq 0$ a unique sheaf

$$\tilde{C}^{\mathrm{op}} \rightarrow \mathrm{CAlg}((n+1)\mathrm{Pr}) = \mathrm{CAlg}(1\mathrm{Pr})_{n\mathrm{Pr}/} : X \mapsto n\mathrm{Pr}_X.$$

For $n = 0$, this is just the usual derived ∞ -category of quasicoherent sheaves on X , at least when X is a countable colimit of affines.¹⁶ For any stack X , we thus get a Stefanich ring

$$(\mathcal{O}(X), D(X), \mathrm{Pr}_X, 2\mathrm{Pr}_X, \dots).$$

For free, this comes with pullback and tensor functoriality; and by presentability, these admit right adjoints (on the level of underlying $(\infty, 1)$ -categories). For $f : Y \rightarrow X$, we denote by

$$f_n^* : n\mathrm{Pr}_X \rightarrow n\mathrm{Pr}_Y$$

the symmetric monoidal pullback, and by $- \otimes_{n\mathrm{Pr}_X} -$ the tensor product inside $n\mathrm{Pr}_X$. Applying the right adjoint $f_{n+1,*}$ to the unit, we can consider $n\mathrm{Pr}_Y$ as an object of $(n+1)\mathrm{Pr}_X$, and we will often use this abuse of notation. (In the next lecture, we will analyze this more systematically for arbitrary Stefanich rings.)

We would like to extend this to a 6-functor formalism by defining lower-!-functors in some generality, ideally as suitable adjoints of pullback. Usually, this is done for $X \mapsto D(X)$; in this lecture, we will look at $X \mapsto n\mathrm{Pr}_X$ for $n \geq 1$. It turns out that at this higher level, we can allow virtually all maps to be !-able; more precisely, we take the following class of maps:

DEFINITION 5.1. Let $f : Y \rightarrow X$ be a map of fpqc stacks. Then f is bi-countably presented if for any affine $\mathrm{Spec}(A) \rightarrow X$, the fibre product $Y \times_X \mathrm{Spec}(A)$ is a countable colimit of $\mathrm{Spec}(B_i)$ for countable presented A -algebras B_i . Let E be the class of bi-countably presented maps.

It is easy to see that E satisfies the usual conditions (stable under composition, pullback, diagonals). We will often use the following general fact. Let C be any small $(\infty, 1)$ -category with all finite limits. Then the countably presented objects of $\mathrm{PShv}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Ani})$ are stable under finite limits, cf. the proof of Proposition 7.3.

THEOREM 5.2. Consider any map $f : Y \rightarrow X$ in E and $n \geq 1$.

(i) The functor

$$f_n^* : n\mathrm{Pr}_X \rightarrow n\mathrm{Pr}_Y$$

has a left adjoint $f_{n\sharp}$ in $(n+1)\mathrm{Pr}_X$; in particular, $f_{n\sharp}$ commutes with all small colimits, with tensoring, and base change.

(ii) For $n \geq 2$, the object $f_{n\sharp}(1) \in n\mathrm{Pr}_X$ is dualizable.

(iii) For $n \geq 2$, the left adjoint $f_{n\sharp}$ is also the right adjoint f_{n*} of f_n^* , and hence f_{n*} also commutes with small colimits, tensoring, and base change.

Thus, for $n \geq 1$ “all maps are étale”, while for $n \geq 2$ “all maps are proper”.

¹⁵We should restrict to κ -compact objects in C^κ for some κ , or work with small sheaves. In fact, we will usually restrict to sheaves that are countable colimits of affines, which are automatically small.

¹⁶In general, there is a difference between taking the limit in $1\mathrm{Pr} = \mathrm{Pr}_{\aleph_1}^L$ or Pr^L , where usually one does the latter.

PROOF. We first assume that $X = \operatorname{Spec}(A)$ is affine, and prove (i) and (ii). By assumption $Y = \operatorname{colim}_{i \in I} \operatorname{Spec}(B_i)$ is a countable colimit of affine $Y_i = \operatorname{Spec}(B_i)$ where each B_i is a countably presented A -algebra. We consider the functor

$$f_n^* : n\operatorname{Pr}_X = n\operatorname{Pr}_A \rightarrow n\operatorname{Pr}_Y = \lim_{i \in I \circ \mathbb{P}} n\operatorname{Pr}_{B_i}.$$

As $n \geq 1$ and all maps are countably presented, we know from the previous lecture that all occurring pullback functors admit linear left adjoints. In particular, the limit defining $n\operatorname{Pr}_Y$ is along right adjoint functors, and hence agrees with the colimit along the left adjoint functors,

$$n\operatorname{Pr}_Y = \operatorname{colim}_{i \in I, \sharp} n\operatorname{Pr}_{B_i}.$$

(We use countability of I here to ensure that this happens in our $\aleph_1$ -presentable setting already.) In this colimit description, we get an evident map

$$n\operatorname{Pr}_Y = \operatorname{colim}_{i \in I, \sharp} n\operatorname{Pr}_{B_i} \xrightarrow{f_{n\sharp}} n\operatorname{Pr}_A$$

(a map in $(n+1)\operatorname{Pr}_A = \operatorname{Mod}_{n\operatorname{Pr}_A}(1\operatorname{Pr})$), as the colimit of the lower-sharp maps $n\operatorname{Pr}_{B_i} \rightarrow n\operatorname{Pr}_A$, which gives the desired left adjoint (cf. again [HY17, Corollary 1.3]). As it is a map in $(n+1)\operatorname{Pr}_A$, it must commute with all small colimits and tensoring; but as any functor of 2-categories preserves adjoints, it must also commute with base change. Note that the formation of $n\operatorname{Pr}_Y \in (n+1)\operatorname{Pr}_A$ also commutes with base change, by the above formula as a colimit. For (ii), we note that evaluating $f_{n\sharp}(1)$, and using that in the affine case $f_{n\sharp} = f_{n*}$, we get

$$f_{n\sharp}(1) = \operatorname{colim}_{i \in I, *} (n-1)\operatorname{Pr}_{B_i}$$

where the colimit is along the lower- $*$ -functors (as these gave rise, by ambidexterity, to the identification between $g_{n\sharp} = g_{n*}$ in the affine case). But for $n \geq 2$ (thus $n-1 \geq 1$), the lower- $*$ -functors $g_{n-1,*}$ are also left adjoints, so this is a countable colimit of dualizable objects along left adjoint functors, and hence itself dualizable by Lemma 4.11.

Now for general X , we get these left adjoints locally in $(n+1)\operatorname{Pr}_X = \lim_{\operatorname{Spec}(A) \rightarrow X} (n+1)\operatorname{Pr}_A$, commuting with base change. This is a limit in $1\operatorname{Pr} = \operatorname{Pr}_{\aleph_1}^L \cong \operatorname{Rex}_{\aleph_1}$, so at the level of $(\infty, 1)$ -categories, we have

$$(n+1)\operatorname{Pr}_X^{\aleph_1} = \lim_{\operatorname{Spec}(A) \rightarrow X} (n+1)\operatorname{Pr}_A^{\aleph_1}.$$

But all objects considered above are $\aleph_1$ -compact. In particular, the object $n\operatorname{Pr}_Y \in (n+1)\operatorname{Pr}_X^{\aleph_1}$ commutes with base change, as it does so in the affine case, and then also the left adjoint descends from the affine case. Part (ii) can be checked after pullback to affines.

Finally, for (iii), we use ambidexterity again. It suffices to see that the unit and counit of the adjunction between $f_{n\sharp}$ and f_n^* admit right adjoints. For the counit, this means that the functor $f_{n\sharp} f_n^* \rightarrow \operatorname{id}$ admits a right adjoint; by linearity, this is equivalent to $f_{n\sharp}(1) \rightarrow 1$ admitting a right adjoint. But by (ii), this is equivalent to the dual map $1 \rightarrow f_{n*}(1)$ admitting a left adjoint, and this is true for $n \geq 2$ (as this is the map $(n-1)\operatorname{Pr}_X \rightarrow (n-1)\operatorname{Pr}_Y$ in $n\operatorname{Pr}_X$). The case of the unit follows similarly by arguing using the diagonal, as in the last lecture. $\square$

COROLLARY 5.3. *Let $f : Y \rightarrow X$, $f' : Y' \rightarrow X$ be bi-countably presented maps. For $n \geq 2$, the objects*

$$(n-1)\operatorname{Pr}_Y, (n-1)\operatorname{Pr}_{Y'} \in n\operatorname{Pr}_X$$

are dualizable, and the Künneth formula

$$(n-1)\operatorname{Pr}_Y \otimes_{n\operatorname{Pr}_X} (n-1)\operatorname{Pr}_{Y'} \cong (n-1)\operatorname{Pr}_{Y \times_X Y'}$$

holds true (where the functor is the exterior tensor product).

In particular, we can determine what sheaves of presentable $(n - 1)$ -categories are on a fibre product in terms of what they are on the factors, and what sheaves of presentable n -categories are on the base. We regard this as the first genuinely surprising statement about quasicoherent sheaves of higher categories: Both parts of this corollary are false at lower levels. Namely, it is absurd that in general $\mathcal{O}(Y) \in D(X)$ should be dualizable (i.e., a perfect complex), or that $\mathcal{O}(Y) \otimes_{D(X)} \mathcal{O}(Y') \cong \mathcal{O}(Y \times_X Y')$ without strong assumptions on Y and Y' . One level up, it is already the case that often $D(Y) \in 1\mathrm{Pr}_X$ is dualizable and the Künneth formula holds true (cf. e.g. [Gai15]) but again this needs some assumption that “ Y is a geometric stack” (over X) in some sense. Going up one more level, these results are (essentially) always true. However, we cannot stay on any fixed categorical level, say $1\mathrm{Pr}$, as the Künneth formula always needs the next categorical level on X .

PROOF. This is direct consequence of the theorem and general properties of six-functor formalisms. $\square$

In fact, to end on a slightly technical note, one can slightly generalize the condition “bi-countably presented” for some of these assertions.

COROLLARY 5.4. *Let $f : Y \rightarrow X$ be a map of $\aleph_1$ -compact objects of $\mathrm{Shv}_{\mathrm{fpqc}}(C)$ (i.e., both X and Y are countable colimits of affine schemes).*

For $n \geq 2$, the right adjoint

$$f_{n*} : n\mathrm{Pr}_Y \rightarrow n\mathrm{Pr}_X$$

of f_n^ commutes with small colimits, tensor products, and pullback. In particular, for $f' : Y' \rightarrow X$ satisfying the same hypotheses, the Künneth formula*

$$(n - 1)\mathrm{Pr}_Y \otimes_{n\mathrm{Pr}_X} (n - 1)\mathrm{Pr}_{Y'} \cong (n - 1)\mathrm{Pr}_{Y \times_X Y'}$$

holds true.

If X is affine, this means that the right adjoint f_{n*} is a map in $\mathrm{Mod}_{n\mathrm{Pr}_A}(\mathrm{Pr}^L)$, but not in general in $\mathrm{Mod}_{n\mathrm{Pr}_A}(\mathrm{Pr}_{\aleph_1}^L)$, i.e. it only fails to preserve $\aleph_1$ -compact objects.

PROOF. By descent, we can assume that X is affine. (Note that as X is a countable colimit of affines $X_i = \mathrm{Spec}(A_i)$, one has

$$n\mathrm{Pr}_X = \lim_I n\mathrm{Pr}_{A_i}$$

not only in $1\mathrm{Pr} = \mathrm{Pr}_{\aleph_1}^L$, but also in Pr^L and thus in $(\infty, 1)$ -categories.) Now $\mathrm{Shv}_{\mathrm{fpqc}}(C_{/X})^{\aleph_1}$ is $\mathrm{Pro}_{\aleph_1}$ of $\mathrm{Shv}_{\mathrm{fpqc}}(C_{/X}^{\aleph_1})^{\aleph_1}$, so f is an $\aleph_1$ -cofiltered limit of bi-countably presented $f_i : Y_i \rightarrow X$. Then $n\mathrm{Pr}_Y$ is the $\aleph_1$ -filtered colimit of the $n\mathrm{Pr}_{Y_i}$ along pullback (as $\aleph_1$ -filtered colimits commute with the countable colimit defining $n\mathrm{Pr}_{Y_i}$ and $n\mathrm{Pr}_Y$ via descent, as we work in the $\aleph_1$ -compactly generated $(\infty, 1)$ -category $1\mathrm{Pr}$). The right adjoint f_{n*} is then correspondingly expressed through an $\aleph_1$ -filtered colimit of the right adjoints $f_{i,n*}$ of $f_{i,n}^* : n\mathrm{Pr}_X \rightarrow n\mathrm{Pr}_{Y_i}$. The properties of commuting with small colimits, tensor products, and pullbacks, are all stable under $(\aleph_1$ -filtered) colimits, so follow for f_{n*} . $\square$

COROLLARY 5.5. *The functor*

$$\mathrm{Shv}_{\mathrm{fpqc}}(C)^{\aleph_1} \rightarrow \mathrm{StRing}_{\mathbb{S}}^{\mathrm{op}} : X \mapsto (\mathcal{O}(X), D(X), 1\mathrm{Pr}_X, 2\mathrm{Pr}_X, \dots)$$

commutes with countable colimits and finite limits.

In other words, if the target were an ∞ -topos, this functor would have all the properties of a morphism of ∞ -topoi (except that we restrict to countable colimits instead of all small colimits).

PROOF. The commutation with countable colimits holds by definition. To prove that it commutes with finite limits, we reduce to the initial object (which is true by definition, working with $\text{StRing}_{\mathbb{S}/}$) and pullbacks. Thus, let $Y \rightarrow X \leftarrow Y'$ be a pullback diagram of $(\aleph_1\text{-compact})$ fpqc stacks. A priori, the tensor product of Stefanich rings amounts to

$$(\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y'), D(Y) \otimes_{D(X)} D(Y'), 1\text{Pr}_Y \otimes_{1\text{Pr}_X} 1\text{Pr}_{Y'}, \dots).$$

This comes with natural maps from one term to endomorphisms of the unit of the next, but these are not yet isomorphisms; we need to “spectrify”. As we will explain in the next lecture, a better approximation to the spectrification is

$$(\mathcal{O}(Y) \otimes_{D(X)} \mathcal{O}(Y'), D(Y) \otimes_{1\text{Pr}_X} D(Y'), 1\text{Pr}_Y \otimes_{2\text{Pr}_X} 1\text{Pr}_{Y'}, \dots).$$

(In other words, in say the zeroth spot, we are forming tensor products not over $\mathcal{O}(X)$, i.e. in $\mathcal{O}(X)$ -modules, but instead in $D(X)$, noting that this is finer as there is a functor $\text{Mod}_{\mathcal{O}(X)}(D(\mathbb{S})) \rightarrow D(X)$.) By Corollary 5.4, starting from $1\text{Pr}_Y \otimes_{2\text{Pr}_X} 1\text{Pr}_{Y'}$, this agrees with

$$(\mathcal{O}(Y \times_X Y'), D(Y \times_X Y'), 1\text{Pr}_{Y \times_X Y'}, \dots)$$

and the latter is already a Stefanich ring. As the first few terms do not matter in a Stefanich ring, this gives the desired result. $\square$

REMARK 5.6. A general slogan in this area is that “everything becomes affine under sufficient categorification”. The results above, that Künneth formulas always hold from the level of 1Pr onwards, justify this in some form. However, the slogan is also a bit imprecise: As discussed in the first lecture, stacks like $B^n\mathbb{G}_m$ are n -affine (in a sense made precise in the next lecture) but not k -affine for $k < n$; so in particular $\bigsqcup B^n\mathbb{G}_m$ is not n -affine for any n . In fact, we will later see that even the infinite-dimensional affine space $\bigcup_{n \geq 0} \mathbb{A}^n$ is not n -affine for any n , although all terms are 0-affine, and the transition maps are cohomologically smooth (as complete intersections are cohomologically smooth in the quasicohherent six-functor formalism). In the next lecture, we will introduce general properties of maps of Stefanich rings, and isolate the conditions that are, in fact, virtually always satisfied.

6. Lecture VI: Stefanich Rings

In this lecture, we start to investigate the presentable $(\infty, 1)$ -category StRing . Starting from this lecture, all results are joint work with Stefanich.

Let us recall the definition:

DEFINITION 6.1. The $\aleph_1$ -presentable $(\infty, 1)$ -category StRing is

$$\begin{aligned} \text{StRing} &= \text{colim}_{1\text{Pr}} (\text{CAlg}(\text{Ani}) \hookrightarrow \text{CAlg}(1\text{Pr}) \hookrightarrow \text{CAlg}(2\text{Pr}) \hookrightarrow \dots) \\ &= \text{lim}_{\text{CAT}_\infty} (\text{CAlg}(\text{Ani}) \leftarrow \text{CAlg}(1\text{Pr}) \leftarrow \text{CAlg}(2\text{Pr}) \leftarrow \dots). \end{aligned}$$

We write $A = (A_0, A_1, A_2, \dots) \in \text{StRing}$ where $A_n \in \text{CAlg}(n\text{Pr})$. Thus, for the Stefanich ring associated to an fpqc stack X , we have $A_n = (n-1)\text{Pr}_X$ for $n \geq 1$.¹⁷

The unit object in StRing is $(*, \text{Ani}, 1\text{Pr}, 2\text{Pr}, \dots)$. In particular, any Stefanich ring A is equipped with compatible symmetric monoidal maps $(n-1)\text{Pr} \rightarrow A_n$. For $n \geq 1$, these can also be seen as coming from

$$A_n \in \text{CAlg}(n\text{Pr}) = \text{CAlg}(\text{Mod}_{(n-1)\text{Pr}}(1\text{Pr})) = \text{CAlg}(1\text{Pr})_{(n-1)\text{Pr}/}.$$

A map $f : A \rightarrow B$ of Stefanich rings is a compatible collection of maps $A_n \rightarrow B_n$ in $\text{CAlg}(n\text{Pr})$ for all $n \geq 1$. In order to keep consistent notation with the previous lecture, we denote these functors as

$$f_{n-1}^* : A_n \rightarrow B_n.$$

Beware the index shift, which matches $A_n = (n-1)\text{Pr}_X$. Also beware that despite notation, on the level of rings f_{n-1}^* is covariant.¹⁸

As f_{n-1}^* , for $n \geq 1$, is a colimit-preserving functor of presentable $(\infty, 1)$ -categories, it admits a right adjoint

$$f_{n-1,*} : B_n \rightarrow A_n.$$

A priori, $f_{n-1,*}$ has no good properties – it is not colimit-preserving or linear. Still, as a right adjoint of a symmetric monoidal functor, it is automatically lax symmetric monoidal, and hence induces a functor

$$f_{n-1,*} : \text{CAlg}(B_n) \rightarrow \text{CAlg}(A_n)$$

on the level of algebras. In particular, the unit $1_{B_n} \in \text{CAlg}(B_n)$ gives a commutative algebra $(B/A)_{n-1} \in \text{CAlg}(A_n)$. Under the forgetful functor $\text{CAlg}(A_n) \rightarrow \text{CAlg}((n-1)\text{Pr})$ (right adjoint to the symmetric monoidal functor $(n-1)\text{Pr} \rightarrow A_n$), this maps to $B_{n-1} \in \text{CAlg}((n-1)\text{Pr})$. In other words, if $B \in \text{StRing}_{A/}$, then for any $n \geq 0$ the object $B_n \in \text{CAlg}(n\text{Pr})$ refines to an object $(B/A)_n \in \text{CAlg}(A_{n+1})$. More precisely, we have the following translation between an internal and an external perspective on A -algebras in Stefanich rings:

PROPOSITION 6.2. *Let A be a Stefanich ring. For any $n \geq 1$, there is a fully faithful functor*

$$\text{CAlg}(A_n) \rightarrow \text{StRing}_{A/}$$

¹⁷The term A_0 is only remembered as a (multiplicative) commutative monoid in anima here. As we will see below, when working over some base algebra, e.g. $\mathbb{S}$, it can be canonically refined, e.g. to $\text{CAlg}(D(\mathbb{S}))$, in particular remembering additive structure.

¹⁸Another justification for the index is that if C is an $(\infty, n+1)$ -category, then objects of C are roughly (∞, n) -categories, morphisms between objects of C are roughly $(\infty, n-1)$ -categories, etc., so our indices refer to the categorical level of the object.

left adjoint to the functor $B \mapsto (B/A)_{n-1}$ constructed above. This induces an equivalence

$$\begin{aligned} \text{StRing}_{A/} &\cong \text{colim}_{1\text{Pr}} (\text{CAlg}(A_1) \hookrightarrow \text{CAlg}(A_2) \hookrightarrow \dots) \\ &= \text{lim}_{\text{CAT}_\infty} (\text{CAlg}(A_1) \leftarrow \text{CAlg}(A_2) \leftarrow \dots), \end{aligned}$$

taking $B \in \text{StRing}_{A/}$ to $((B/A)_0, (B/A)_1, \dots)$.

PROOF. Tautologically,

$$\text{StRing}_{A/} = \text{lim}_{\text{CAT}_\infty} (\text{CAlg}(\text{Ani})_{A_0/} \leftarrow \text{CAlg}(1\text{Pr})_{A_1/} \leftarrow \dots).$$

But for $n \geq 1$,

$$\text{CAlg}(n\text{Pr})_{A_n/} = \text{CAlg}(\text{Mod}_{(n-1)\text{Pr}}(1\text{Pr}))_{A_n/} = \text{CAlg}(1\text{Pr})_{A_n/} = \text{CAlg}(\text{Mod}_{A_n}(1\text{Pr})),$$

where in the middle, we switch from considering the under-category under A_n from $A_n \in \text{CAlg}(n\text{Pr}) = \text{CAlg}(1\text{Pr})_{(n-1)\text{Pr}/}$ to $A_n \in \text{CAlg}(1\text{Pr})$. Under this equivalence, the transition functors are given by the composite

$$\text{CAlg}(\text{Mod}_{A_{n+1}}(1\text{Pr})) \rightarrow \text{CAlg}(A_{n+1}) \rightarrow \text{CAlg}(\text{Mod}_{A_n}(1\text{Pr})),$$

where the first is the general adjunction between $\text{CAlg}(\mathcal{V})$ and $\text{CAlg}(\text{Mod}_{\mathcal{V}}(1\text{Pr}))$, and the second is the adjunction between $\text{Mod}_{A_n}(1\text{Pr})$ and A_{n+1} coming from the structure of A as a Stefanich ring. In particular, the above limit is equivalent to the limit

$$\text{StRing}_{A/} \cong \text{lim}_{\text{CAT}_\infty} (\text{CAlg}(A_1) \leftarrow \text{CAlg}(A_2) \leftarrow \dots);$$

it is clear from the construction that this takes, at each level, B to $(B/A)_{n-1} \in \text{CAlg}(A_n)$. All transition functors here admit left adjoints, which are given by the composite

$$\text{CAlg}(A_n) \rightarrow \text{CAlg}(\text{Mod}_{A_n}(1\text{Pr})) \rightarrow \text{CAlg}(A_{n+1}).$$

It remains to see that these composite functors $\text{CAlg}(A_n) \rightarrow \text{CAlg}(A_{n+1})$ are fully faithful. As everything in sight is happening in $1\text{Pr} = \text{Pr}_{\aleph_1}^L$, it suffices to check this on $\aleph_1$ -compact objects. Given any countably presented $R \in \text{CAlg}(A_n)$, we get $\text{Mod}_R(A_n) \in \text{CAlg}(\text{Mod}_{A_n}(1\text{Pr}))$. The pullback functor $F_R : A_n \rightarrow \text{Mod}_R(A_n)$ admits an A_n -linear right adjoint $G_R : \text{Mod}_R(A_n) \rightarrow A_n$ (the forgetful functor). As R is countably presented, this also preserves $\aleph_1$ -compact objects and hence G_R is a right adjoint of F_R in $\text{Mod}_{A_n}(1\text{Pr})$. As $f : \text{Mod}_{A_n}(1\text{Pr}) \rightarrow A_{n+1}$ is a functor of 2-categories, it follows that $f(G_R)$ is the right adjoint of $f(F_R)$. The right adjoint to $\text{CAlg}(A_n) \rightarrow \text{CAlg}(A_{n+1})$ takes $f(\text{Mod}_R(A_n))$ to the endomorphisms of the unit, as an object of $\text{CAlg}(A_n)$. But this is given by the image of the unit in A_n under $f(G_R) \circ f(F_R) = f(G_R \circ F_R) = f(R \otimes -)$. But f induces an equivalence $A_n \cong \text{End}_{A_{n+1}}(1)$, so this takes the unit in A_n to $R \in A_n = \text{End}_{A_{n+1}}(1)$. Thus, the unit of the adjunction between $\text{CAlg}(A_n)$ and $\text{CAlg}(A_{n+1})$ is an equivalence on $\aleph_1$ -compact objects, finishing the proof of fully faithfulness. $\square$

In the proof, we showed that the composite functor

$$\text{CAlg}(A_n) \rightarrow \text{CAlg}(\text{Mod}_{A_n}(1\text{Pr})) \rightarrow \text{CAlg}(A_{n+1})$$

is fully faithful. The first functor is also fully faithful. However, the second functor is usually not fully faithful. Still, it is fully faithful on dualizable objects:

LEMMA 6.3. *For any Stefanich ring A and $n \geq 0$, the functor*

$$\mathrm{Mod}_{A_n}(1\mathrm{Pr}) \rightarrow A_{n+1}$$

is fully faithful on dualizable objects. In fact, it is even fully faithful on $\aleph_1$ -filtered colimits of dualizable objects.

PROOF. Let $F : \mathrm{Mod}_{A_n}(1\mathrm{Pr}) \rightarrow A_{n+1}$ denote the given functor, with right adjoint G . As F preserves $\aleph_1$ -compact objects, G preserves $\aleph_1$ -filtered colimits. It is then enough to check fully faithfulness on dualizable objects. But if $F : \mathcal{C} \rightarrow \mathcal{D}$ is any symmetric monoidal functor of symmetric monoidal categories, then fully faithfulness on dualizable objects follows from $1 \rightarrow G(1)$ being an isomorphism. Indeed, given $X, Y \in \mathcal{C}$ dualizable, one has

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) = \mathrm{Hom}_{\mathcal{D}}(F(X \otimes Y^*), 1) = \mathrm{Hom}_{\mathcal{C}}(X \otimes Y^*, G(1)).$$

If $G(1) = 1$, this is $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. But in our situation $1 \rightarrow G(1)$ is an isomorphism by assumption. $\square$

6.1. Shifting. We note that for any Stefanich ring $A = (A_1, A_2, \dots)$ (starting from $A_1 \in \mathrm{CAlg}(1\mathrm{Pr})$ for this discussion), also the shift $(A_2, A_3, \dots)$ is a Stefanich ring. Indeed, recall that for $n \geq 1$ there are forgetful functors

$$\mathrm{CAlg}((n+1)\mathrm{Pr}) \rightarrow \mathrm{CAlg}(n\mathrm{Pr}).$$

Taking into account the effect on the unit, shifting is an equivalence:

PROPOSITION 6.4. *The shifting functor $(A_1, A_2, \dots) \mapsto (A_2, A_3, \dots)$ induces an equivalence*

$$\mathrm{StRing} \cong \mathrm{StRing}_{(1\mathrm{Pr}, 2\mathrm{Pr}, \dots)}/.$$

In particular, shifting n times, for any $n \geq 0$, induces an equivalence

$$\mathrm{StRing} \cong \mathrm{StRing}_{(n\mathrm{Pr}, (n+1)\mathrm{Pr}, \dots)}/.$$

PROOF. This follows from

$$\mathrm{CAlg}((n+1)\mathrm{Pr}) = \mathrm{CAlg}(n\mathrm{Pr})_{n\mathrm{Pr}}/.$$

More precisely, $\mathrm{CAlg}((n+1)\mathrm{Pr}) = \mathrm{CAlg}(\mathrm{Mod}_{n\mathrm{Pr}}(1\mathrm{Pr})) = \mathrm{CAlg}(1\mathrm{Pr})_{n\mathrm{Pr}}/$. But we also have $\mathrm{CAlg}(n\mathrm{Pr}) = \mathrm{CAlg}(1\mathrm{Pr})_{(n-1)\mathrm{Pr}}/$, and considering $n\mathrm{Pr}$ as an object in there, one has

$$\mathrm{CAlg}(n\mathrm{Pr})_{n\mathrm{Pr}}/ = (\mathrm{CAlg}(1\mathrm{Pr})_{(n-1)\mathrm{Pr}}/)_{n\mathrm{Pr}}/ = \mathrm{CAlg}(1\mathrm{Pr})_{n\mathrm{Pr}}/ = \mathrm{CAlg}((n+1)\mathrm{Pr}). \quad \square$$

If we would have worked stably, i.e. with $\mathrm{StRing}_{\mathbb{S}}/$ (where $\mathbb{S}$ denotes $(D(\mathbb{S}), 1\mathrm{Pr}_{\mathrm{st}}, \dots)$), then this shifting operation would not exist.

We do not currently have a good intuitive or “geometric” understanding of this shifting operation. In fact, its very existence is linked to our precise choice of foundations; it is not a priori clear that a functor $2\mathrm{Pr} \rightarrow 1\mathrm{Pr}$ should exist. This functor is also not compatible with changing κ (which we fixed to be $\aleph_1$). However, at a technical level, the shifting operation allows us to assume without loss of generality that $n = 0$ in many of the following discussions on properties of morphisms.

6.2. Tensor products of Stefanich rings. Limits of Stefanich rings are easy to compute, as they are just limits at each level. Colimits, however, are more subtle to compute. The reason is that if one performs the colimit at each level, the resulting object will be a sequence of $D_n \in \mathbf{CAlg}(n\mathbf{Pr})$ equipped with maps

$$D_n \rightarrow \mathrm{End}_{D_{n+1}}(1),$$

but these usually fail to be isomorphisms. One has to universally enforce that these are equivalences, i.e. “spectrify” (thinking of Stefanich rings as analogous to infinite loop spaces $X_0 \cong \Omega X_1 \cong \Omega^2 X_2 \dots$, also known as spectra). By the adjoint functor theorem, this can always be done. Concretely, one replaces such a sequence $(D_n)_{n \geq 0}$ by a new sequence

$$D'_n = \mathrm{End}_{D_{n+1}}(1),$$

with a map $D_n \rightarrow D'_n$, and then repeats, using transfinite induction (and using filtered colimits at limit stages). As $D \mapsto \mathrm{End}_D(1)$ commutes with $\aleph_1$ -filtered colimits, after $\aleph_1$ many steps one will be done.

Fortunately, in practice there will be more concrete ways of computing the tensor products. We will particularly be interested in pushouts, i.e. tensor products. Thus, consider a pushout diagram $B \xleftarrow{f} A \xrightarrow{g} C$ of Stefanich rings. The above recipe is to form

$$(B_0 \otimes_{A_0} C_0, B_1 \otimes_{A_1} C_1, B_2 \otimes_{A_2} C_2, \dots)$$

and then spectrify. Under the equivalence of Proposition 6.2, we can instead form the tensor product in $\mathrm{StRing}_{A/}$, which leads us to form

$$((B/A)_0 \otimes_{A_1} (C/A)_0, (B/A)_1 \otimes_{A_2} (C/A)_1, (B/A)_2 \otimes_{A_3} (C/A)_2, \dots),$$

and then spectrify.¹⁹ In the previous lecture, we have in fact already used this alternative perspective in the proof of Corollary 5.5. There, it turned out that the spectrification is almost unnecessary: It changes only the first two terms.

In a yet more refined way, we can also think of $B \otimes_A C$ as the image of $B \in \mathrm{StRing}_{A/}$ under $\mathrm{StRing}_{A/} \xrightarrow{g^*} \mathrm{StRing}_{C/}$. This leads us to form

$$(g_0^*(B/A)_0, g_1^*(B/A)_1, g_2^*(B/A)_2, \dots),$$

and then spectrify.²⁰ We will repeatedly use this perspective on base change in the following; in particular, we will isolate general classes of maps for which the resulting spectrification is almost unnecessary in the sense that it changes only finitely many terms.

6.3. Affine maps. Proposition 6.2 suggests the following definition.

DEFINITION 6.5. Let $A \in \mathrm{StRing}$ and $n \geq 0$. An A -algebra $B \in \mathrm{StRing}_{A/}$ is n -affine if it lies in the essential image of

$$\mathbf{CAlg}(A_{n+1}) \rightarrow \mathrm{StRing}_{A/}.$$

¹⁹There is some inconsistency in the subscript of a tensor: In the last display it denotes the category in which one forms the tensor product. The previous display should then probably be more verbosely written as $(B_0 \otimes_{\mathrm{Mod}(A_0)} C_0, \dots)$.

²⁰At least in contexts like this one, the indexing of g_n^* seems reasonable.

Equivalently, for all $m \geq n$, the functor

$$\mathrm{Mod}_{A_{m+1}}(1\mathrm{Pr}) \rightarrow A_{m+2}$$

takes $\mathrm{Mod}_{(B/A)_m}(A_{m+1})$ to $(B/A)_{m+1}$, so B is determined by $(B/A)_n \in \mathrm{CAlg}(A_{n+1})$ by iteratively passing to module categories (in the A -internal sense). Note also that by Lemma 6.3, the functor $\mathrm{Mod}_{A_{m+1}}(1\mathrm{Pr}) \rightarrow A_{m+2}$ is fully faithful on $(\aleph_1\text{-filtered colimits of})$ dualizables, and $\mathrm{Mod}_{(B/A)_m}(A_{m+1}) \in \mathrm{Mod}_{A_{m+1}}(1\mathrm{Pr})$ is (an $\aleph_1\text{-filtered colimit of})$ dualizable(s). Indeed, if $R \in \mathrm{CAlg}(A_{m+1})$ is countable presented, then $\mathrm{Mod}_R(A_{m+1})$ is dualizable.

PROPOSITION 6.6. *Let $A \in \mathrm{StRing}$ and $n \geq 0$. The class of n -affine $B \in \mathrm{StRing}_{A/}$ is stable under all small colimits. Moreover:*

- (i) *If B is n -affine over A , then for all $m \geq n$ the functor*

$$\mathrm{Mod}_{(B/A)_m}(A_{m+1}) \rightarrow B_{m+1}$$

is an equivalence of $(\infty, 1)$ -categories (hence, in $\mathrm{Mod}_{A_{m+1}}(1\mathrm{Pr})$).

- (ii) *If $g : A \rightarrow C$ is any map of Stefanich rings, then for any n -affine A -algebra B , also $B \otimes_A C = g^*B$ is n -affine over C , and for all $m \geq n$, $g_m^*(B/A)_m \cong (B \otimes_A C/C)_m$.*
- (iii) *Consider composable maps $f : A \rightarrow B$ and $g : B \rightarrow C$ with composite $h : A \rightarrow C$. Assume that f is n -affine. Then g is n -affine if and only if h is n -affine.*

In particular, n -affine maps are stable under composition, base change, and passage to diagonals.

WARNING 6.7. The converse to (i) is wrong. Consider the map of stacks $f : * \rightarrow B^2\mathbb{G}_m$ (over a field k , say), and let $A \rightarrow B$ be the corresponding map of Stefanich rings. Then f is 1-affine, so (i) is satisfied for $m \geq 1$. But it is also satisfied for $m = 0$ as $A_1 = B_1 = D(k)$ and $(B/A)_0 = k$. But f cannot be 0-affine, as $(B/A)_0 = k$ is the unit.

WARNING 6.8. In the literature, e.g. [Gai15], the notion of 1-affineness is often only a statement at the level of 2-categories, and does not include a similar assertion at all higher levels. The main reason was, of course, that these higher levels were not even considered. For the general formalism, it is better to use this stronger condition, which is also satisfied in the examples of 1-affine maps in the literature.

PROOF. Stability under small colimits is clear, as $\mathrm{CAlg}(A_{n+1}) \rightarrow \mathrm{StRing}_{A/}$ preserves small colimits.

For part (i), if B is n -affine, then for all $m \geq n$, one can write $(B/A)_{m+1}$ as the image of $\mathrm{Mod}_{(B/A)_m}(A_{m+1})$ under $\mathrm{Mod}_{A_{m+1}}(1\mathrm{Pr}) \rightarrow A_{m+2}$. But the latter functor is fully faithful on the subcategory containing the unit and $\mathrm{Mod}_{(B/A)_m}(A_{m+1})$ by Lemma 6.3, so the underlying $(\infty, 1)$ -category is unchanged, and hence $B_{m+1} = \mathrm{Mod}_{(B/A)_m}(A_{m+1})$.

Part (ii) follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{CAlg}(A_{n+1}) & \hookrightarrow & \mathrm{CAlg}(A_{n+2}) \hookrightarrow \dots \\ \downarrow g_n^* & & \downarrow g_{n+1}^* \\ \mathrm{CAlg}(C_{n+1}) & \hookrightarrow & \mathrm{CAlg}(C_{n+2}) \hookrightarrow \dots \end{array} \quad \begin{array}{c} \mathrm{StRing}_{A/} \\ \downarrow g^* \\ \mathrm{StRing}_{C/}, \end{array}$$

which is pasted from

$$\begin{array}{ccccc} \mathrm{CAlg}(A_{n+1}) & \longrightarrow & \mathrm{CAlg}(\mathrm{Mod}_{A_{n+1}}(1\mathrm{Pr})) & \longrightarrow & \mathrm{CAlg}(A_{n+2}) \\ \downarrow g_n^* & & \downarrow & & \downarrow g_{n+1}^* \\ \mathrm{CAlg}(C_{n+1}) & \longrightarrow & \mathrm{CAlg}(\mathrm{Mod}_{C_{n+1}}(1\mathrm{Pr})) & \longrightarrow & \mathrm{CAlg}(C_{n+2}). \end{array}$$

Here the commutativity of the first square is the general assertion that module categories base change; while the commutativity of the second square follows from the commutative diagram of symmetric monoidal $(\infty, 1)$ -categories

$$\begin{array}{ccc} \mathrm{Mod}_{A_{n+1}}(1\mathrm{Pr}) & \longrightarrow & A_{n+2} \\ \downarrow & & \downarrow g_{n+1}^* \\ \mathrm{Mod}_{C_{n+1}}(1\mathrm{Pr}) & \longrightarrow & C_{n+2}, \end{array}$$

which comes from the datum of the map g of Stefanich rings.

For part (iii), we note that if B is n -affine over A , then $\mathrm{StRing}_{B/}$ maps isomorphically to

$$\mathrm{StRing}_{A/} \times_{\mathrm{CAlg}(A_{n+1})} \mathrm{CAlg}(A_{n+1})_{(B/A)_n/}.$$

In other words, C is fully determined by the map $A \rightarrow C$ plus a map $(B/A)_n \rightarrow (C/A)_n$ in $\mathrm{CAlg}(A_{n+1})$. Now the $(\infty, 1)$ -category of n -affine C over B is given by $\mathrm{CAlg}(B_{n+1})$, while the $(\infty, 1)$ -category of B -algebras C that are n -affine over A is given by $\mathrm{CAlg}(A_{n+1})_{(B/A)_n/}$. By (i), we have $\mathrm{CAlg}(B_{n+1}) \cong \mathrm{CAlg}(A_{n+1})_{(B/A)_n/}$, compatibly with the above interpretation. $\square$

6.4. Proper maps. In Lecture IV, we saw that 0-affine maps admit many well-behaved right adjoints. We are abstracting this into the following notion.²¹

DEFINITION 6.9. Let $f : A \rightarrow B$ be a map of Stefanich rings, and let $n \geq 0$. Then f is n -prim if for all $m \geq n + 1$, the functor

$$1 \xrightarrow{f_{m-1}^*} (B/A)_m$$

admits a right adjoint $f_{m-1,*} : (B/A)_m \rightarrow 1$ in A_{m+1} .

In other words, the external right adjoint $f_{m-1,*} : B_m \rightarrow A_m$ refines to an internal right adjoint in A_{m+1} .

PROPOSITION 6.10. *Let A be a Stefanich ring and $n \geq 0$.*

- (i) *If $B \in \mathrm{StRing}_{A/}$ is n -affine and countably presented, then it is n -prim.*
- (ii) *The collection of n -prim B in $\mathrm{StRing}_{A/}$ is stable under all filtered colimits. Moreover, the formation of filtered colimits of n -prim Stefanich A -algebras commutes with the functor $\mathrm{StRing}_{A/} \rightarrow \mathrm{CAlg}(A_m)$ for $m \geq n$.*
- (iii) *If $f : A \rightarrow B$ is n -prim and $g : A \rightarrow C$ is any map of Stefanich rings, then the base change $f' : C \rightarrow B \otimes_A C$ is n -prim, and the map*

$$g_m^*(B/A)_m \rightarrow (B \otimes_A C/C)_m$$

²¹The name is taken from a recently introduced similar name in the theory of six functors, cf. e.g. [HM24] or [Sch25b], used there to describe maps for which pullback has an internal right adjoint.

in $\text{CAlg}(A_{m+1})$ is an isomorphism for $m \geq n$. More precisely, the formation of f_{m*} commutes with base change for $m \geq n$.

- (iv) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be composable maps with composite $h : A \rightarrow C$. If f and g are n -prim, then h is n -prim.

In particular, n -prim maps are stable under base change and composition. However, they are not stable under passage to diagonals. For example, for a field k of characteristic 0 (for simplicity) and a variety X over k , its de Rham stack X_{dR} is 0-prim over k , but the diagonal is not.

PROOF. By shifting, we may assume (for notational convenience only) that $n = 0$.

For (i), let B be 0-affine over A and countably presented. This ensures that $(B/A)_0 \in \text{CAlg}(A_1)$ is countably presented, and then by induction all $(B/A)_m \in \text{CAlg}(A_{m+1})$ are countably presented. For $m \geq 1$, the pullback functor

$$A_m \xrightarrow{f_{m-1}^*} \text{Mod}_{(B/A)_{m-1}}(A_m)$$

has a right adjoint $f_{m-1,*}$ in $\text{Mod}_{A_m}(\mathbf{1Pr})$. Taking the image under $\text{Mod}_{A_m}(\mathbf{1Pr}) \rightarrow A_{m+1}$, and using that by 0-affineness, this sends $\text{Mod}_{(B/A)_{m-1}}(A_m)$ to $(B/A)_m$, this shows that

$$1 \xrightarrow{f_m^*} (B/A)_m$$

has a right adjoint $f_{m-1,*}$ in A_{m+1} , as desired.

For (ii), let I be some countable filtered index category and $i \mapsto B(i)$ a diagram in 0-prim Stefanich A -algebras. Let

$$(B/A)_m := \text{colim}_{i \in I} (B(i)/A)_m \in \text{CAlg}(A_{m+1})$$

for $m \geq 0$. As I is filtered, the colimit commutes with the forgetful functor $\text{CAlg}(A_{m+1}) \rightarrow A_{m+1}$. We need to see that this is already a Stefanich ring. By Lemma 6.11 below, the unit map

$$1 \xrightarrow{f_m^*} (B/A)_{m+1} = \text{colim}_{i \in I} (B(i)/A)_{m+1}$$

admits a right adjoint f_{m*} in A_{m+2} , whose composite with f_m^* is the colimit of $f(i)_{m*} f(i)_m^*$, using the individual right adjoints $f(i)_{m*} : (B(i)/A)_{m+1} \rightarrow 1$, which we assumed to exist. We want to compute $\text{End}_{(B/A)_{m+1}}(1)$ as an object of $\text{CAlg}(A_{m+1})$. This is equivalently given by the composite $f_{m*} f_m^* \in \text{End}_{A_{m+2}}(1) = A_{m+1}$. But by the description of $f_{m*} f_m^*$ as the colimit of the $f(i)_{m*} f(i)_m^*$, this yields precisely $\text{colim}_{i \in I} (B(i)/A)_m$, as desired. Once we know that the colimit in $\text{StRing}_{A/}$ is indeed given by B with $(B/A)_m$ as defined above, the above argument already proved that this is 0-prim.

For (iii), we similarly want to show that the

$$g_m^*(B/A)_m \in \text{CAlg}(C_{m+1})$$

form a Stefanich ring. Note that for $m \geq 0$, the adjunction $1 \xrightarrow{f_m^*} (B/A)_{m+1} \xrightarrow{f_{m*}} 1$ in A_{m+2} yields an adjunction

$$1 \xrightarrow{g_{m+1}^*(f_m^*)} g_{m+1}^*(B/A)_{m+1} \xrightarrow{g_{m+1}(f_{m*})} 1$$

in C_{m+2} . Computing $\text{End}_{g_{m+1}^*(B/A)_{m+1}}(1)$ as an object of $\text{CAlg}(C_{m+1})$ is then again equivalent to understanding the composite of the unit map $1 \rightarrow g_{m+1}^*(B/A)_{m+1}$ with its right adjoint; this

yields an object of $C_{m+1} = \text{End}_{C_{m+2}}(1)$ which is the underlying object of $\text{End}_{g_{m+1}^*(B/A)_{m+1}}(1) \in \text{CAlg}(C_{m+1})$. But we have just identified this right adjoint, and hence this composite is given by

$$g_{m+1}^*(f_{m*}f_m^*) \in \text{End}_{C_{m+2}}(1).$$

But on $\text{End}_-(1)$, g_{m+1}^* reduces to $g_m^* : A_{m+1} \rightarrow C_{m+1}$, while $f_{m*}f_m^*$ corresponds to $\text{End}_{(B/A)_{m+1}}(1) = (B/A)_m \in A_{m+1}$. Thus, the previous display evaluates to $g_m^*(B/A)_m \in C_{m+1}$, as desired. Now the argument also proves that the base change is 0-prim. In fact, it also identifies the right adjoint of $g_{m+1}^*(f_m^*)$ with $g_{m+1}^*(f_{m*})$, which means that the formation of the right adjoint f_{m*} commutes with base change.

For (iv), if $f : A \rightarrow B$ and $g : B \rightarrow C$ are 0-prim, then note that for $m \geq 1$ the forgetful functor $B_{m+1} \rightarrow A_{m+1}$ preserves right adjoints, and hence the right adjoint of $1 \xrightarrow{g_{m+1}^*} (C/B)_m$ in B_{m+1} yields a right adjoint of $(B/A)_m \xrightarrow{g_m^*} (C/A)_m$ in A_{m+1} . Thus, composing right adjoints for

$$1 \xrightarrow{f_{m+1}^*} (B/A)_m \xrightarrow{g_m^*} (C/A)_m$$

yields the claim. $\square$

We used the following lemma about general presentable $(\infty, 2)$ -categories.

LEMMA 6.11. *Let $\mathbb{C} \in 2\text{Pr}$. Let I be a countable filtered index category with an initial object $0 \in I$, and let $X : I \rightarrow \mathbb{C}$ be a functor. Assume that for all $i \in I$, the map $F_i : X_0 \rightarrow X_i$ admits a right adjoint $F_i^R : X_i \rightarrow X_0$ in $\mathbb{C}$. Then*

$$F : X_0 \rightarrow \text{colim}_I X$$

admits a right adjoint F^R , given by the compatible collection of functors

$$\text{colim}_{j \in I_i} F_j^R \circ (X_i \rightarrow X_j) : X_i \rightarrow X_0.$$

In particular, the composite $F^R F$ is the colimit $\text{colim}_{i \in I} F_i^R F_i$ in $\text{End}_{\mathbb{C}}(X_0)$, and the unit is the map $\text{id} \rightarrow \text{colim}_{i \in I} F_i^R F_i$ which is the colimit of the maps $\text{id} \rightarrow F_i^R F_i$.

PROOF. We use Yoneda. More precisely, we can first assume that X_0 and all X_i are 1-atomic, by replacing $\mathbb{C}$ by the 1Pr-linear $(\infty, 1)$ -category that is 1-atomically generated by the subcategory of X_0 and all X_i . Then if X is any of these 1-atomic objects, the functor $\text{Hom}_{\mathbb{C}}(X, -) : \mathbb{C} \rightarrow 1\text{Pr}$ is 1Pr-linear. A morphism $F : Y \rightarrow Y'$ in $\mathbb{C}$ that is right adjoint will map to a right adjoint functor in 1Pr under $\text{Hom}_{\mathbb{C}}(X, -)$. Conversely, if right adjoints exist under any evaluation, and moreover for any map $X \rightarrow X'$, these right adjoints are preserved under the corresponding transformation $\text{Hom}_{\mathbb{C}}(X', -) \rightarrow \text{Hom}_{\mathbb{C}}(X, -)$ (i.e., the Beck–Chevalley condition), then F admits a right adjoint. As usual, this follows by applying the hypothesis to $X = Y$ and $X = Y'$. The Beck–Chevalley condition in fact follows from the claimed explicit description.

Thus, we can assume $\mathbb{C} = 1\text{Pr}$. In this case, this is a standard fact. Alternatively, one can also write down the unit and counit of the adjunction explicitly, and check that they satisfy the triangles. $\square$

As the notion of n -prim is not stable under passing to diagonals, the following strengthening is more useful.

DEFINITION 6.12. Let $f : A \rightarrow B$ be a map of Stefanich rings, and let $n \geq 0$. Then f is n -proper if all iterated diagonals $f : A \rightarrow B$, $\Delta_f : B \otimes_A B \rightarrow B$, $\Delta_{\Delta_f} : B \otimes_{B \otimes_A B} B \rightarrow B$, ... are n -prim.

PROPOSITION 6.13. *Let A be a Stefanich ring and $n \geq 0$.*

- (i) *If $B \in \text{StRing}_{A/}$ is n -affine and countably presented, then it is n -proper.*
- (ii) *The collection of n -proper B in $\text{StRing}_{A/}$ is stable under all countable colimits.*
- (iii) *If $f : A \rightarrow B$ is n -proper and $g : A \rightarrow C$ is any map of Stefanich rings, then the base change $f' : C \rightarrow B \otimes_A C$ is n -proper.*
- (iv) *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be composable maps with composite $h : A \rightarrow C$. If f is n -proper, then g is n -prim (resp. n -proper) if and only if h is n -prim (resp. n -proper).*

PROOF. Most of this follows directly from Proposition 6.10. Part (i), for example, follows from part (i) there, plus stability of n -affine maps under passing to diagonals. The case of filtered colimits in part (ii) similarly follows from part (ii) there. It then remains to handle relative tensor products. Those follow from parts (iii) and (iv), using in particular the cancellation property in part (iv).

Part (iii) follows immediately from part (iii) for prim maps. For part (iv), note that diagonals of a composite map can be written as composites of base changes of diagonals of the individual maps; hence stability under composition for proper maps follows from the same result for prim maps and stability under composition. For the converse, assume that f is n -proper and h is n -prim (resp. n -proper). Write g as the composite $B \rightarrow B \otimes_A C \rightarrow C$ where the first map is a base change of $h : A \rightarrow C$ and hence n -prim (resp. n -proper), and the second map is a base change of $\Delta_f : B \otimes_A B \rightarrow B$ and hence n -proper. Thus, the composite g is n -prim (resp. n -proper). $\square$

Being 0-proper is closely related to being rigid in the sense of Gaitsgory–Rozenblyum [GR17, Chapter I, Definition 9.1.2].

PROPOSITION 6.14. *Let A be a Stefanich ring and let B be a countably presented 1-affine Stefanich A -algebra. Then B is 0-proper over A if and only if $(B/A)_1 \in \text{CAlg}(A_2)$ is rigid in A_2 , i.e.:*

- (i) *The unit map $1 \rightarrow (B/A)_1$ admits a right adjoint in A_2 , and*
- (ii) *The multiplication map $(B/A)_1 \otimes_{A_2} (B/A)_1 \rightarrow (B/A)_1$ admits a right adjoint in*

$$\text{Mod}_{(B/A)_1 \otimes_{A_2} (B/A)_1}(A_2).$$

PROOF. Recall that B is automatically 1-proper, so being 0-proper amounts to having in addition right adjoints at the lowest level. If B is 0-proper, then (i) and (ii) are satisfied by applying the assumption of being 0-prim to $A \rightarrow B$ and its diagonal $B \otimes_A B \rightarrow B$, noting that by Proposition 6.6 (i), $(B \otimes_A B)_2 = \text{Mod}_{(B/A)_1 \otimes_{A_2} (B/A)_1}(A_2)$.

For the converse, it suffices to see that (i) and (ii) imply that the diagonal $B \otimes_A B \rightarrow B$ is 0-affine: Indeed, then the diagonal is certainly 0-proper, and together with $A \rightarrow B$ itself being 0-prim shows that $A \rightarrow B$ is 0-proper. But the diagonal is 1-affine, 0-prim, and admits a section. This implies that it is 0-affine by the next lemma. $\square$

The following lemma is often useful to reduce the level of affineness.

LEMMA 6.15. *Let $f : A \rightarrow B$ be a map of Stefanich rings. Assume that f is $n + 1$ -affine, n -prim, and there is some $n + 1$ -affine map $g : B \rightarrow A$ with $fg = \text{id}_B$. Then f is n -affine.*

PROOF. By shifting, we can assume $n = 0$. Replacing A by the image of $(B/A)_0 \in \text{CAlg}(A_1) \rightarrow \text{StRing}_{A/}$, we can assume that $(B/A)_0 = 1 \in \text{CAlg}(A_1)$. In this case, we have to see that $A \rightarrow B$ is an isomorphism; by 1-affineness, it suffices to see that $1 \rightarrow (B/A)_1$ is an isomorphism. Now

$f_0^* : 1 \rightarrow (B/A)_1$ has a right adjoint f_{0*} in A_2 , as f is 0-prim. Moreover, the unit $\text{id} \rightarrow f_{0*}f_0^*$ in $A_1 = \text{End}_{A_2}(1)$ is an isomorphism, as this is the map $1 \rightarrow (B/A)_0$. We would now like to argue that as g_0^* splits the map $1 \rightarrow (B/A)_1$, it must be an isomorphism. But that is not quite true. What is true is that after applying g_{1*} to $1 \rightarrow (B/A)_1$, we get the map $(A/B)_1 \rightarrow 1$ that is indeed split by $g_0^* : 1 \rightarrow (A/B)_1$. As g_{1*} preserves the fully faithful adjunction, this implies that $1 \rightarrow (B/A)_1$ becomes an isomorphism after applying g_{1*} . But as g is 1-affine, $g_{1*} : A_2 = \text{Mod}_{(A/B)_1}(B_2) \rightarrow B_2$ is conservative, so this finishes the proof. $\square$

6.5. Étale maps. In Lecture V, we saw that in many situations, we have left adjoints to pullback instead; and that by ambidexterity we can pass between the settings of left and right adjoints at the expense of losing a categorical level. Let us thus introduce the analogues of the previous definition for left adjoints.

The analogue of n -prim is the following:²²

DEFINITION 6.16. Let $f : A \rightarrow B$ be a map of Stefanich rings, and let $n \geq 0$. Then f is n -suave if for all $m \geq n + 1$, the functor

$$1 \xrightarrow{f_{m-1}^*} (B/A)_m$$

admits a left adjoint $f_{m-1,\sharp} : (B/A)_m \rightarrow 1$ in A_{m+1} , and moreover the following condition holds. For $m \geq n$, let

$$(B/A)_m^! = f_{m,\sharp}(1) \in A_{m+1},$$

which comes with a natural map $(B/A)_m^! \xrightarrow{f_{m-1,!}} 1$ in A_{m+1} . For $m \geq n + 1$, this map admits a right adjoint $f_{m-1}^!$ in A_{m+1} .

We note that $(B/A)_m^! = f_{m,\sharp}(1)$ is naturally a commutative coalgebra in A_{m+1} , which dualizes to the algebra $(B/A)_m$. In particular, the counit map $(B/A)_m^! \rightarrow 1$ dualizes to the map $1 \rightarrow (B/A)_m$. Thus, the existence of the right adjoint for $(B/A)_m^! \rightarrow 1$ is a strengthening of the existence of the left adjoint of $1 \rightarrow (B/A)_m$ (and we ask both in the same regime $m \geq n + 1$). If $(B/A)_m^!$ is dualizable for all $m \geq n + 1$, the conditions are equivalent; this is how we argued in the last lecture, effectively. One should regard the existence of the right adjoint of $(B/A)_m^! \rightarrow 1$ as being the “correct” condition to put, but its mere formulation requires the existence of the left adjoint one level higher.

REMARK 6.17. The notation $(B/A)_m^!$ is suggestive of !-sheaves: Any fpqc stack X is 1-suave over the point $\text{Spec}(\mathbb{S})$, and hence one gets an object $(X/\mathbb{S})_1^! \in 1\text{Pr}_{\mathbb{S}}$; explicitly, this can be computed to be the category of !-sheaves, i.e. one takes the limit of $D(A)$, for affine A , along !-pullback. For $m \geq 2$, ambidexterity shows that in this example $(B/A)_m^! = (B/A)_m$, and this will be generally the case for us – only at the lowest level there will (usually) be a difference between $(B/A)_m$ and $(B/A)_m^!$.

Under such an identification $(B/A)_m^! = (B/A)_m$, the functor denoted $f_{m-1,!}$ above is an incarnation of the “usual” !-functors in six-functor formalisms. We will make this point precise later.

We have an analogue of Proposition 6.10, at least of parts (iii) and (iv). There is also an analogue of (ii), but it deals with limits rather than colimits, and we will discuss this in the next lecture.

²²Again, the name is chosen in analogy with a similar notion for six functors.

PROPOSITION 6.18. *Let A be a Stefanich ring and $n \geq 0$.*

- (i) *If $f : A \rightarrow B$ is n -suave and $g : A \rightarrow C$ is any map of Stefanich rings, then the base change $f' : C \rightarrow B \otimes_A C$ is n -suave, and the map*

$$g_m^*(B/A)_m^! \rightarrow (B \otimes_A C/C)_m^!$$

in C_{m+1} is an isomorphism for $m \geq n$. More precisely, the formation of $f_{m\sharp}$ commutes with base change, for $m \geq n$.

- (ii) *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be composable maps with composite $h : A \rightarrow C$. If f and g are n -suave, then h is n -suave.*

PROOF. By shifting, we assume $n = 0$. For (i), we first want to show that the

$$(g_m^*(B/A)_m^!)^\vee \in \text{CAlg}(C_{m+1})$$

form a Stefanich ring, where $-^\vee$ denotes the dual (in the sense of the internal mapping object to the unit) in C_{m+1} .

For $m \geq 0$, consider the adjunction $1 \xrightarrow{f_{m-1}^!} (B/A)_m^! \xrightarrow{f_{m-1,!}} 1$ in A_{m+1} . This dualizes, under internal mapping to the unit, to the adjunction $1 \xrightarrow{f_{m-1}^*} (B/A)_m \xrightarrow{f_{m-1,\sharp}} 1$. In particular, the composite maps $1 \rightarrow 1$ are the same, and hence given by $(B/A)_{m-1}^! = f_{m-1,\sharp}(1) \in A_m = \text{End}_{A_{m+1}}(1)$.

If we apply g_m^* to the adjunction $1 \rightarrow (B/A)_m^! \rightarrow 1$, we get an adjunction

$$1 \xrightarrow{g_m^*(f_{m-1}^!)} g_m^*(B/A)_m^! \xrightarrow{g_m^*(f_{m-1,!})} 1$$

in C_{m+1} . The composite map $1 \rightarrow 1$ is given by $g_{m-1}^*(B/A)_{m-1}^! \in C_m = \text{End}_{C_{m+1}}(1)$. If we dualize the displayed adjunction, we get an adjunction

$$1 \rightarrow (g_m^*(B/A)_m^!)^\vee \rightarrow 1$$

in C_{m+1} , whose composite is still given by $g_{m-1}^*(B/A)_{m-1}^! \in C_m$. Here $(g_m^*(B/A)_m^!)^\vee$ is a commutative algebra in C_{m+1} , and the first map is the unit map, while the second is its left adjoint. It follows that the composite is a predual to $\text{End}_{(g_m^*(B/A)_m^!)^\vee}(1)$. Thus,

$$\text{End}_{(g_m^*(B/A)_m^!)^\vee}(1) = (g_{m-1}^*(B/A)_{m-1}^!)^\vee$$

for $m \geq n + 1$, showing that indeed $(g_m^*(B/A)_m^!)^\vee$, for $m \geq n$, forms a Stefanich C -algebra. This is also the correct base change. Indeed, let us form instead $(g_m^*(B/A)_m)_{m \geq n}$, and compute the endomorphism of the unit. To do so, start with the adjunction for $m \geq 1$

$$1 \xrightarrow{f_{m-1}^*} (B/A)_m \xrightarrow{f_{m-1,\sharp}} 1,$$

with induced endomorphism of 1 again being $(B/A)_{m-1}^!$, and take its pullback under $g_m^* : A_{m+1} \rightarrow C_{m+1}$. This shows that there is a similar adjunction

$$1 \rightarrow g_m^*(B/A)_m \rightarrow 1$$

with induced endomorphism of 1 being $g_{m-1}^*(B/A)_{m-1}^!$; and this yields a predual of $\text{End}_{g_m^*(B/A)_m}(1)$. Thus,

$$\text{End}_{g_m^*(B/A)_m}(1) = (g_{m-1}^*(B/A)_{m-1}^!)^\vee$$

for $m \geq 1$, thereby showing that indeed $(g_m^*(B/A)_m^!)^\vee$, for $m \geq n$, is the correct base change. The base-changed adjunctions used in the proof above then immediately yield all the required left and right adjoints, and also show that the formation of these adjoints commutes with base change.

For (ii), if $f : A \rightarrow B$ and $g : B \rightarrow C$ are 0-suave, then note that for $m \geq 1$ the forgetful functor $B_{m+1} \rightarrow A_{m+1}$ preserves left adjoints, and hence the left adjoint of $1 \xrightarrow{g_{m-1}^*} (C/B)_m$ in B_{m+1} yields a left adjoint of $(B/A)_m \xrightarrow{g_{m-1}^*} (C/A)_m$ in A_{m+1} . Thus, composing left adjoints for

$$1 \xrightarrow{f_{m-1}^*} (B/A)_m \xrightarrow{g_{m-1}^*} (C/A)_m$$

gives the existence of the left adjoint. We can then define the diagram

$$(C/A)_m^! \rightarrow (B/A)_m^! \rightarrow 1$$

and again both individual maps admit right adjoints, hence so does their composite. $\square$

Again, being n -suave is not stable under diagonals, so often the following strengthening is more useful:

DEFINITION 6.19. Let $f : A \rightarrow B$ be a map of Stefanich rings, and let $n \geq 0$. Then f is n -étale if all iterated diagonals $f : A \rightarrow B$, $\Delta_f : B \otimes_A B \rightarrow B$, $\Delta_{\Delta_f} : B \otimes_{B \otimes_A B} B \rightarrow B$, ... are n -suave.

PROPOSITION 6.20. Let A be a Stefanich ring and $n \geq 0$.

- (i) If $f : A \rightarrow B$ is n -étale and $g : A \rightarrow C$ is any map of Stefanich rings, then the base change $f' : C \rightarrow B \otimes_A C$ is n -étale.
- (ii) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be composable maps with composite $h : A \rightarrow C$. If f is n -étale, then g is n -suave (resp. n -étale) if and only if h is n -suave (resp. n -étale).

PROOF. This follows directly from Proposition 6.18, arguing as in the proof of Proposition 6.13. $\square$

6.6. Ambidexterity. In the previous lectures, we proved that all (bi-countably presented) fpqc stacks are 1-étale and 2-proper. Abstracting the argument there, we have the following.

PROPOSITION 6.21. Let $f : A \rightarrow B$ be a map of Stefanich rings and $n \geq 0$.

- (i) If f and Δ_f are n -prim, then f is $n+1$ -suave. In particular, if f is n -proper, then f is $n+1$ -étale.
- (ii) If f and Δ_f are n -suave, then f is $n+1$ -prim. In particular, if f is n -étale, then f is $n+1$ -proper.

Moreover, under either assumption of (i) or (ii), one gets an identification $f_{m\sharp} \cong f_{m*}$ for $m \geq n+1$, and this functor preserves dualizable objects.

PROOF. By shifting, we may assume $n = 0$. For part (i), we claim that for $m \geq 1$, the map $f_m^* : 1 \rightarrow (B/A)_{m+1}$ in A_{m+2} has a right adjoint f_{m*} such that the unit and counit of the adjunction admit right adjoints. Namely, the unit is given by $1 \xrightarrow{f_{m-1}^*} (B/A)_m$ in A_{m+1} so is handled by f being 0-prim, while the counit is a map in $\text{End}_{A_{m+2}}((B/A)_{m+1})$. There is a functor $(B \otimes_A B)_{m+1} \rightarrow \text{End}_{A_{m+2}}((B/A)_{m+1})$ given as usual by kernels; this takes the map $1 \rightarrow (B/B \otimes_A B)_m$ in $(B \otimes_A B)_{m+1}$ to the counit of the adjunction. In particular, if this admits a right adjoint in $(B \otimes_A B)_{m+1}$, then we get the desired right adjoint of the counit. Now Proposition 4.4 shows that for $m \geq 1$, the right adjoint f_{m*} is also a left adjoint $f_{m\sharp}$. In particular, $f_{m*} = f_{m\sharp}$

preserves dualizable objects by Lemma 6.22 below. Hence $f_{m\sharp}(1)$ is dualizable for $m \geq 1$; this shows that the extra condition in Definition 6.16 is also true.

Part (ii) is similar. More precisely, we claim that for $m \geq 1$ the unit and counit of the adjunction between f_m^* and $f_{m\sharp}$ admit right adjoints. For the counit, this is part of the definition of being 0-suave for f . For the unit, it follows from $(B/B \otimes_A B)_m^! \rightarrow 1$ having a right adjoint in $(B \otimes_A B)_{m+1}$, under the functor $(B \otimes_A B)_{m+1} \rightarrow \text{End}_{A_{m+2}}((B/A)_{m+1})$ given by kernels (using left adjoints). $\square$

The following lemma shows in particular that being both n -prim and n -suave ensures that $f_{m\sharp}$ and f_{m*} preserve dualizable objects for $m \geq n$.

LEMMA 6.22. *Let $f^* : C \rightarrow D$ be a map in $\text{CAlg}(\text{Pr}^L)$. Assume that f^* has both a C -linear left adjoint $f_\sharp$ and a C -linear right adjoint f_* . Then $f_\sharp$ and f_* preserve dualizable objects.*

PROOF. For any dualizable $N \in D$ and any $M \in C$, we have

$$\begin{aligned} \text{Hom}_C(f_\sharp N, 1) \otimes M &\cong f_* \text{Hom}_D(N, 1) \otimes M \\ &\cong f_*(\text{Hom}_D(N, 1) \otimes f^* M) \\ &\cong f_* \text{Hom}_D(N, f^* M) \\ &\cong \text{Hom}_C(f_\sharp N, M), \end{aligned}$$

showing that $f_\sharp N$ is dualizable. By the computation, the dual is automatically $f_* N^\vee$ where $N^\vee$ is the dual of N in D . This also shows that f_* preserves dualizable objects. $\square$

7. Lecture VII: Gestalten

With the definition from the last lecture, we can finally state the theorem from the first lecture in a precise form.

THEOREM 7.1. *Let A be a Stefanich ring A and $n \geq 0$. Consider the $(\infty, 1)$ -category $\mathrm{StRing}_{A/}^{n\text{-}\acute{\text{e}}\text{t}}$ of n -étale A -algebras. Then $(\mathrm{StRing}_{A/}^{n\text{-}\acute{\text{e}}\text{t}})^{\mathrm{op}}$ is a countably $(\infty, 1)$ -topos, i.e.*

$$\mathrm{Ind}_{\aleph_1}(\mathrm{StRing}_{A/}^{n\text{-}\acute{\text{e}}\text{t}})^{\mathrm{op}}$$

is an $(\infty, 1)$ -topos.

The qualifier “for all practical purposes” of the first lecture thus refers to two aspects: On the one hand, to a restriction to countable colimits instead of all small colimits. This is a set-theoretic nuisance that has to be there as StRing and its opposite cannot both be presentable. If we replace $\aleph_1$ by a different uncountable regular cardinal, one would similarly replace “countable” above.

The slightly more serious aspect is the restriction to n -étale Stefanich rings, for some fixed n . However, as for example all algebraic stacks are 1-étale, and there are extremely strong stability properties of n -étale Stefanich rings (possibly up to a slight increase of n by 1 or 2), in practice one can safely just fix $n = 10$ at the start and proceed. As one increases n , any reasonable construction, for a very generous notion of “reasonable”, will stabilize at some very early level.

7.1. Countably $(\infty, 1)$ -topoi. Let us recall the definition of $(\infty, 1)$ -topoi from [Lur09b], and then describe its countable variant that will be relevant to us.

THEOREM 7.2 ([Lur09b, Theorem 6.1.0.6]). *Let C be a presentable $(\infty, 1)$ -category. The following conditions are equivalent.*

- (i) *There is a small $(\infty, 1)$ -category S and a left exact accessible localization*

$$\mathrm{PShv}(S) = \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Ani}) \rightarrow C.$$

- (ii) *Taking κ so that C is κ -presentable and C^κ is stable under finite limits, the functor*

$$\mathrm{PShv}(C^\kappa) \rightarrow C$$

of presentable $(\infty, 1)$ -categories, extending the embedding $C^\kappa \subset C$, commutes with finite limits.

- (iii) *Giraud’s axioms are satisfied, i.e.*

- (a) *Colimits are universal.*
- (b) *Coproducts are disjoint.*
- (c) *Groupoid objects are effective.*

We briefly recall the meaning of Giraud’s axioms:

- (a) Colimits are universal: This means that for all maps $f : Y \rightarrow X$, the pullback $f^* : C_{/X} \rightarrow C_{/Y}$ commutes with small colimits.
- (b) Coproducts are disjoint: This means that for all $X, Y \in C$, the fibre product $X \times_{X \sqcup Y} Y$ is empty (i.e., the initial object).
- (c) Groupoid objects are effective: This is the most subtle and important axiom, and the one which differs between the $(1, 1)$ -categorical and the $(\infty, 1)$ -categorical version of topoi. A

groupoid object in an $(\infty, 1)$ -category C is a simplicial object $X_\bullet$ such that for all $n \geq 1$, the natural map

$$X_n \rightarrow X_1^{\times n/X_0}$$

(induced by the n maps $X_n \rightarrow X_1$ over X_0) is an isomorphism. Examples of groupoid objects are Čech nerves of maps $f : X \rightarrow Y$, where $X_0 = X$, $X_1 = X \times_Y X$, and in general $X_n = X^{\times(n+1)/Y}$; if $X \rightarrow Y$ is also an effective epimorphism, such groupoid objects are called effective. In this case, the “effective epimorphism” condition ensures that Y is necessarily the geometric realization of $X_\bullet$, and the condition of effectivity becomes equivalent to asking that the resulting map $X_1 \rightarrow X_0 \times_Y X_0$ is an isomorphism, cf. [Lur09b, Proposition 6.1.2.11].

Note that even if all X_i are 0-truncated, typically the geometric realization is only 1-truncated – one gets classifying anima of groups this way.

This suggests the following countable version.

PROPOSITION 7.3. *Let C be a small $(\infty, 1)$ -category admitting all countable colimits and finite limits. The following conditions are equivalent.*

- (i) *There is small $(\infty, 1)$ -category S with finite limits such that there is a left exact localization*

$$\mathrm{PShv}(S)^{\aleph_1} \rightarrow C.$$

- (ii) *The functor*

$$\mathrm{PShv}(C)^{\aleph_1} \rightarrow C$$

(commuting with countable colimits, and extending the identity $C \rightarrow C$) commutes with finite limits.

- (iii) *Giraud’s axioms are satisfied in C , i.e.*

- (a) *Countable colimits are universal.*
- (b) *Coproducts are disjoint.*
- (c) *Groupoid objects are effective.*

- (iv) *The $\aleph_1$ -presentable $(\infty, 1)$ -category $\mathrm{Ind}_{\aleph_1}(C)$ is an $(\infty, 1)$ -topos.*

We note that (b) and especially the most important condition (c) only refer to countable colimit diagrams, so the “essence” of being an $(\infty, 1)$ -topos is contained in the countable colimits.

PROOF. Implicit in (i) (and (ii)) is that $\mathrm{PShv}(S)^{\aleph_1} \subset \mathrm{PShv}(S)$ is stable under finite limits. General finite limits reduce to fibre products. If $X, Y, Z \in \mathrm{PShv}(S)^{\aleph_1}$ with maps $X \rightarrow Y \leftarrow Z$, then to show that $X \times_Y Z$ is countably presented, we can first write X and Z as countable colimits to reduce to the case $X, Z \in S$. We can write Y as a geometric realization of a simplicial object $Y_\bullet$ consisting of countable disjoint unions of objects of S . The maps $X, Z \rightarrow Y$ factor over Y_0 , and $X \times_Y Z = X \times_{Y_0} (Y_0 \times_Y Y_0) \times_{Y_0} Z$, so we can further reduce to $X = Z = Y_0$. Now we can apply Kan’s iterated simplicial subdivision Ex_∞ to make $Y_\bullet$ into a Kan complex; this keeps countable presentation. Then $Y_0 \times_Y Y_0$ is also the geometric realization of $Y_0 \times_{Y_i} Y_0$, which is still countably presented.

It is clear that (ii) implies (i). If (iv) is satisfied, then (ii) and (iii) are satisfied by Theorem 7.2, taking $\kappa = \aleph_1$. But (i) implies (iv). $\square$

DEFINITION 7.4. A countably $(\infty, 1)$ -topos is a small $(\infty, 1)$ -category C satisfying the conditions of Proposition 7.3.

7.2. Limits of n -suave Stefanich rings. Before starting the proof of Theorem 7.1, let us note that the statement is surprising. Namely, stating that in the opposite of a category of A -algebras, colimits are universal, translates into *limits* of A -algebras commuting with base change. In fact, the following statements are true:

- (i) For any countable category I and any diagram $I \rightarrow \text{StRing}_{A/}^{n\text{-ét}} : i \mapsto B_i$ of n -étale A -algebras, also the limit $B = \lim_{i \in I} B_i$ is n -étale over A .
- (ii) Moreover, given any map $g : A \rightarrow C$, the base change $- \otimes_A C$ preserves the limits from (i).

At the level of usual commutative rings, such statements are very false. It is thus a bit surprising that they become true at this categorified level. The key phenomenon is of course ambidexterity, which blurs the distinction between left and right adjoints, and correspondingly between limits and colimits.

As a first step towards the proof of Theorem 7.1, we prove (i) and (ii) in the n -suave case.

PROPOSITION 7.5. *Let $A \in \text{StRing}$ and $n \geq 0$.*

- (i) *The collection of n -suave B in $\text{StRing}_{A/}$ is stable under all countable limits. Moreover, the formation of countable limits of n -suave Stefanich A -algebras commutes with the contravariant functor $B \mapsto (B/A)_m^! \in A_{m+1}$ for $m \geq n$, i.e. takes countable limits to colimits.*
- (ii) *For any $g : A \rightarrow C$, the functor $- \otimes_A C : \text{StRing}_{A/} \rightarrow \text{StRing}_{C/}$ commutes with countable limits of n -suave Stefanich algebras.*

PROOF. As usual, we shift to assume $n = 0$. For (i), let I be some countable index category and $i \mapsto B(i)$ a diagram in 0-suave Stefanich A -algebras. As forming limits of Stefanich rings is done naively, the limit B is given by the collection of

$$(B/A)_m := \lim_{i \in I} (B(i)/A)_m \in \text{CAlg}(A_{m+1})$$

for $m \geq 0$. By Lemma 7.6 below, for $m \geq 1$ the map $1 \rightarrow (B/A)_m$ admits a left adjoint, computed as the colimit of the left adjoints of the individual $1 \rightarrow (B(i)/A)_m$. Thus, we see in particular that

$$(B/A)_m^! = \text{colim}_{i \in I^{\text{op}}} (B(i)/A)_m^!$$

as commutative coalgebras in A_{m+1} . Now we need to see that for $m \geq 1$ the map $(B/A)_m^! \rightarrow 1$ admits a right adjoint. This is also part of Lemma 7.6 below.

Now part (ii) follows from base change of n -suave algebras $A \rightarrow B$ being computed by taking the pullback $g_m^*(B/A)_m^!$ of the coalgebra $(B/A)_m^!$ and passing to the dual afterwards; this operation commutes with base change, as

$$(B/A)_m^! = \text{colim}_{i \in I^{\text{op}}} (B(i)/A)_m^!$$

by (i). □

We used another lemma about (co)limits and adjoints in presentable $(\infty, 2)$ -categories.

LEMMA 7.6. *Let $\mathbb{C} \in 2\text{Pr}$. Let I be a countable index category and $X : I \rightarrow \mathbb{C}$ some diagram.*

- (i) *Assume that $Y \in \mathbb{C}$ is equipped with a map $F : Y \rightarrow \lim_I X$ such that for all $i \in I$, the composite map $F_i : Y \rightarrow X_i$ admits a left adjoint F_i^L . Then F admits a left adjoint F^L , given as the composite*

$$\lim_I X \rightarrow \lim_I^{\text{lax}} X \xrightarrow{\lim_I^{\text{lax}} F_i} \lim_I^{\text{lax}} Y = \text{Fun}(I^{\text{op}}, Y) \xrightarrow{\text{colim}_{I^{\text{op}}}} Y,$$

so that $F^L F$ is given by the colimit $\operatorname{colim}_{i \in I^{\text{op}}} F_i^L F_i$, with counit $F^L F \rightarrow \text{id}$ the corresponding colimit $\operatorname{colim}_{i \in I^{\text{op}}} F_i^L F_i \rightarrow \text{id}$ of counits.

- (ii) Assume that $Y \in \mathbb{C}$ is equipped with a map $F : \operatorname{colim}_I X \rightarrow Y$ such that for all $i \in I$, the composite map $F_i : X_i \rightarrow Y$ admits a right adjoint F_i^R . Then F admits a right adjoint F^R , given as the composite

$$Y \xrightarrow{((X_i \rightarrow \operatorname{colim}_I X) \circ F_i^R)_i} \operatorname{Fun}(I, \operatorname{colim}_I X) \xrightarrow{\operatorname{colim}_I} \operatorname{colim}_I X,$$

so that FF^R is given by the colimit $\operatorname{colim}_{i \in I} F_i F_i^R$, with counit $FF^R \rightarrow \text{id}$ given as the corresponding colimit of counits.

PROOF. The proof is very similar to the proof of Lemma 6.11; see also [HY17, Theorem B] for (i). $\square$

7.3. Proof of Theorem 7.1. Let us now prove Theorem 7.1. At this point, the argument is basically formal, but a bit confusing. For psychological reasons, it is helpful to prove the following generalization.

Assume that C is some small $(\infty, 1)$ -category with finite limits, together with a left exact $\aleph_1$ -accessible localization of $\operatorname{PShv}(C)$, which we will denote as $\operatorname{Shv}(C)$; equivalently, a left exact localization $\operatorname{Shv}(C)^{\aleph_1}$ of $\operatorname{PShv}(C)^{\aleph_1}$. We assume that C is subcanonical, so that C embeds into $\operatorname{Shv}(C)^{\aleph_1}$.

Moreover, let

$$\mathcal{O} : C^{\text{op}} \rightarrow \operatorname{StRing}_{A/}^{n\text{-}\text{et}}$$

be a sheaf of n -étale Stefanich A -algebras on C . We assume that $\mathcal{O}$ takes finite limits in C to finite colimits in $\operatorname{StRing}_{A/}$; i.e., the final object of C goes to A , and fibre products go to tensor products (“the Künneth formula holds”).

By descent, we can extend $\mathcal{O}$ to a functor, still denoted

$$\mathcal{O} : \operatorname{Shv}(C)^{\text{op}} \rightarrow \operatorname{StRing}_{A/}.$$

We will generally restrict attention to the countably $(\infty, 1)$ -topos $\operatorname{Shv}(C)^{\aleph_1}$.

THEOREM 7.7. *In the situation above, the functor*

$$\mathcal{O} : \operatorname{Shv}(C)^{\aleph_1, \text{op}} \rightarrow \operatorname{StRing}_{A/}$$

takes values in n -étale Stefanich A -algebras, and takes finite limits in $\operatorname{Shv}(C)^{\aleph_1}$ to finite colimits in $\operatorname{StRing}_{A/}$.

This generalizes the case where C was $\operatorname{CAlg}(D(\mathbb{S}))^{\aleph_1, \text{op}}$, considered in Lecture V. In fact, as we will see, the proof in the general case is basically identical to the proof in Lecture V.

Let us first show that Theorem 7.7 implies Theorem 7.1.

THEOREM 7.7 IMPLIES THEOREM 7.1. We take $C = (\operatorname{StRing}_{A/}^{n\text{-}\text{et}})^{\text{op}}$, with the trivial Grothendieck topology (i.e., $\operatorname{Shv}(C) := \operatorname{PShv}(C)$). This is small, as for n -étale Stefanich algebras B , all $(B/A)_m \in A_{m+1}$ are dualizable for $m \geq n + 1$, and hence $\aleph_1$ -compact. In this case, Theorem 7.7 says that $\mathcal{O}$ yields a functor

$$\operatorname{PShv}(C) \rightarrow C \subset \operatorname{StRing}_{A/}^{\text{op}}$$

that commutes with finite limits. Thus, C satisfies condition (ii) of Proposition 7.3. Moreover, we note that the proof also shows the next corollary. $\square$

COROLLARY 7.8. *Let $A \in \text{StRing}$ and $n \geq 0$. Any countable limit of n -étale Stefanich A -algebras is itself n -étale over A .*

PROOF. Indeed, the previous proof showed that the limit taken in $\text{StRing}_{A/}$ stays in $\text{StRing}_{A/}^{n\text{-ét}}$, for any countable diagram of n -étale A -algebras. $\square$

Now we prove Theorem 7.7 (noting that in its proof we cannot yet use Corollary 7.8).

PROOF OF THEOREM 7.7. First, we show that for any map $f : Y \rightarrow X$ in $\text{Shv}(C)$, the map $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is n -suave. If X is representable, i.e. lies in $C \subset \text{Shv}(C)$, then this follows from Proposition 7.5 (i). Moreover, Proposition 7.5 (ii) shows that for any map of representables $g : X' \rightarrow X$, the natural map $\mathcal{O}(X') \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \rightarrow \mathcal{O}(X' \times_X Y)$ is an isomorphism. More precisely, for any $m \geq n$, we get the cocommutative coalgebra $(\mathcal{O}(X' \times_X Y)/\mathcal{O}(X'))_m^! \in \mathcal{O}(X')_{m+1}$, and its formation commutes with base change.

Thus, by descent, we can define a cocommutative coalgebra $B_m^! \in \mathcal{O}(X)_{m+1}$ whose base changes to the representable $X' \rightarrow X$ are the $(\mathcal{O}(X' \times_X Y)/\mathcal{O}(X'))_m^! \in \mathcal{O}(X')_{m+1}$; in particular, for $m \geq n+1$ the counit $B_m^! \xrightarrow{g_{m-1,!}} 1$ admits a right adjoint $g_{m-1}^!$ in $\mathcal{O}(X)_{m+1}$. Taking the internal Hom to the unit, its dual is

$$\begin{aligned} \text{Hom}_{\mathcal{O}(X)_{m+1}}(B_m^!, 1) &= \lim_{X' \in C, g: X' \rightarrow X} g_{m*} \text{Hom}_{\mathcal{O}(X')_{m+1}}((\mathcal{O}(X' \times_X Y)/\mathcal{O}(X'))_m^!, 1) \\ &= \lim_{X' \in C, g: X' \rightarrow X} g_{m*}(\mathcal{O}(X' \times_X Y)/\mathcal{O}(X'))_m \\ &= \lim_{X' \in C, g: X' \rightarrow X} (\mathcal{O}(X' \times_X Y)/\mathcal{O}(X))_m \\ &= (\mathcal{O}(Y)/\mathcal{O}(X))_m, \end{aligned}$$

using that Y is the colimit of the $X' \times_X Y$, and hence $(\mathcal{O}(-)/\mathcal{O}(X))_m \in \mathcal{O}(X)_{m+1}$ is the limit. The right adjoint of $B_m^! \rightarrow 1$ then dualizes to a left adjoint of $1 \rightarrow (\mathcal{O}(Y)/\mathcal{O}(X))_m$ for $m \geq n+1$. Moreover, the value of this left adjoint at the unit is $B_{m-1}^! \in \mathcal{O}(X)_m$. Indeed, this value can also

be computed as the composite of $1 \xrightarrow{g_{m-1}^!} B_m^! \xrightarrow{g_{m-1,!}} 1$, and this is $B_{m-1}^!$ (by descent). We have already seen that $B_m^! \rightarrow 1$ has a right adjoint when $m \geq n+1$; thus, $\mathcal{O}(Y)$ is n -suave over $\mathcal{O}(X)$, with $(\mathcal{O}(Y)/\mathcal{O}(X))_m^! = B_m^!$. Moreover, the argument shows that for all representable $X' \rightarrow X$, the map

$$\mathcal{O}(X') \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \rightarrow \mathcal{O}(X' \times_X Y)$$

is an isomorphism; indeed, base change of n -suave maps is computed via base change of the coalgebras, so this follows from $(\mathcal{O}(Y)/\mathcal{O}(X))_m^! = B_m^!$.

By 2-out-of-3 for isomorphisms, it follows now that for any pullback $X' \rightarrow X \leftarrow Y$, the map

$$\mathcal{O}(X') \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \rightarrow \mathcal{O}(X' \times_X Y)$$

is an isomorphism.

At this point, we know that $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is n -suave for all $f : Y \rightarrow X$; and we know that the Künneth formula always holds true. Thus, passing to diagonals can be done either on sheaves, or on Stefanich rings; but as we know suaveness for all maps, we know it in particular for all iterated diagonals, and hence in fact for all $f : Y \rightarrow X$, the map $f^* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is n -étale. $\square$

7.4. Descent. Let $\text{Gest} = \text{StRing}^{\text{op}}$; for any $A \in \text{StRing}$, we denote the corresponding object of Gest by $\text{Gest}(A)$. If A is n -affine, we will also often simply denote this as $\text{Gest}(A_n)$. More generally, if $X = \text{Gest}(A)$ and Y is n -affine over X , associated to $(\mathcal{O}(Y)/\mathcal{O}(X))_n \in \text{CAlg}(\mathcal{O}(X)_{n+1})$, we write

$$Y = \text{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_n).$$

One convenient aspect of the formalism is that the properties of morphisms introduced above are local with respect to the tautological topology. More precisely:

PROPOSITION 7.9. *Let $f : Y \rightarrow X$ be map of Gestalten. Let $g : X' \rightarrow X$ be a map of Gestalten that is an effective epimorphism, i.e. the natural map*

$$\text{colim}_{[n] \in \Delta^{\text{op}}} X'^{\times_X (n+1)} \rightarrow X$$

is an isomorphism; if g is n -étale for some n , this means that $g : X' \rightarrow X$ is a cover in the countably $(\infty, 1)$ -topos $\text{Gest}_{/X}^{n\text{-ét}}$.

Then, for any $n \geq 0$, f is n -affine (resp. n -prim, n -proper, n -suave, n -étale) if and only if $f' = f \times_X X' : Y' = Y \times_X X' \rightarrow X'$ is n -affine (resp. n -prim, n -proper, n -suave, n -étale).

PROOF. In one direction, this follows from base change compatibility of all notions. For the converse direction, assume that f' has one of these properties; we want to show the same for f . But as countable colimits of Gestalten correspond to countable limits of Stefanich rings, and those are computed naively, we have

$$\mathcal{O}(X)_m = \lim_{[n] \in \Delta^{\text{op}}} \mathcal{O}(X'^{\times_X (n+1)})_m$$

for all m . If f' is n -affine, then f gives rise to a commutative algebra object in the displayed category for any $m \geq n$, and these are related by passing to internal module categories; this shows that f is n -affine. If f' is n -prim, then for $m \geq n+1$, the object $f'_{m-1,*}1 \in \mathcal{O}(X')_m$ satisfies base change, so defines an object in the limit $\mathcal{O}(X)_m$, which must then be given by $f_{m-1,*}1$. The map $1 \rightarrow f_{m-1,*}1$ in $\mathcal{O}(X)_m$ admits a right adjoint, as its base changes do. This shows that f is n -prim. The n -proper case follows formally by passing to diagonals; similarly, the n -étale case reduces to the n -suave case. In the n -suave case, one argues similarly using $f'_{m-1,\#}1$. $\square$

While at first sight this may seem formal and unsurprising, this behaviour is in fact much better than could maybe be expected. Namely, already in rigid-analytic geometry, there is no good notion of affine morphism. For example, if X is the 2-dimensional closed unit disc over $\mathbb{Q}_p$ (considered as a Gestalt via taking $\text{Gest}(\mathcal{O}(X))$ over the solid p -adic numbers $\text{Gest}(D(\mathbb{Q}_p, \square))$), and $U \subset X$ is the locus where either $|x_1| \geq \frac{1}{p}$ or $|x_2| \geq \frac{1}{p}$, then U is not affine (as it has cohomology in degree 1). However, one can cover X by the rational open subsets

$$V_1 = \{|x_1|, |x_2| \leq \frac{1}{p^2}\}, V_2 = \{|x_1|, \frac{1}{p^2} \leq |x_2|\}, V_3 = \{|x_2|, \frac{1}{p^2} \leq |x_1|\}.$$

Then $U \times_X V_i \subset V_i$ is affine for all i . Indeed, it is empty for $i = 0$; and $U \times_X V_1 \subset V_1$ (resp. $U \times_X V_2 \subset V_2$) is given by the locus where $|x_2| \geq \frac{1}{p}$ (resp. $|x_1| \geq \frac{1}{p}$) which is affine.

In the formalism of Gestalten, it follows from the previous proposition that U does in fact count as 0-affine, associated to the nonconnective $\mathbb{E}_\infty$ -algebra of global sections $\mathcal{O}(U)$, i.e. $U = \text{Gest}(\mathcal{O}(U))$ over $\text{Gest}(D(\mathbb{Q}_p, \square))$.

8. Lecture VIII: Examples

As in the last lecture, we let $\text{Gest} = \text{StRing}^{\text{op}}$; for any $A \in \text{StRing}$, we denote the corresponding object of Gest by $\text{Gest}(A)$. Generally, we will fix some base A — usually either the initial ring $(*, \text{Ani}, 1\text{Pr}, 2\text{Pr}, \dots)$ or the 1-affine Stefanich ring corresponding to $D_{\geq 0}(k)$ or $D(k)$ for some (connective) E_{∞} -ring k — and work in the (countably) $(\infty, 1)$ -topos $\text{Gest}_{/X}^{n\text{-et}}$ of n -étale Gestalten over the corresponding base Gestalt X . However, all examples discussed below will be 2-étale, and I will generally drop the superscript.

Let us discuss various examples.

8.1. The point. For any $X = \text{Gest}(A)$, the point $* \in \text{Gest}_{/X}$ is given by $X = \text{Gest}(A)$. In particular, in the absolute case, this is $\text{Gest}(*, \text{Ani}, 1\text{Pr}, \dots)$, and is 0-affine, 0-proper, and 0-étale.

8.2. The empty set. Again, for any base X , we have the empty set $\emptyset \in \text{Gest}_{/X}$. This is the Gestalt associated to the zero ring $\emptyset = \text{Gest}(*, *, *, \dots)$. This is always 1-affine and hence 1-proper, and also 0-étale (as a colimit — the empty colimit — of 0-étale maps). Over the absolute base, it is not 0-affine (as the 0-th term for both $*$ and $\emptyset$ is $*$), but it becomes 0-affine after base change to $\text{Gest}(\text{Ani}_*)$ corresponding to the symmetric monoidal presentable $(\infty, 1)$ -category of pointed anima Ani_* . In particular, after base change to $D_{\geq 0}(k)$ or $D(k)$, it is 0-affine.

8.3. Finite sets. For any base $X = \text{Gest}(A_0, A_1, \dots)$ and any finite set I , we have the corresponding object $I = \bigsqcup_{i \in I} * \in \text{Gest}_{/X}$. This is $\text{Gest}_X(\prod_I A_0, \prod_I A_1, \dots)$, and is always 1-affine and hence 1-proper, and also 0-étale. If A_1 is semiadditive, then $I \in \text{Gest}_{/X}$ is 0-affine. It suffices to check this for $I = * \sqcup *$ over $X = \text{Gest}(\text{Mon}_{\mathbb{E}_{\infty}}(\text{Ani}))$, the initial semiadditive symmetric monoidal presentable $(\infty, 1)$ -category. (Indeed, once it holds for the 2-element set, it holds for all powers, and all subsets thereof.) But then $A_1 \times A_1$ is equivalent to $M \in A_1$ plus idempotent maps $e_1 : M \rightarrow M$, $e_2 : M \rightarrow M$ with $e_1 + e_2 = 1$ — the idempotents then cut out retracts M_1 and M_2 such that the map $M_1 \oplus M_2 \rightarrow M$ is an isomorphism. In particular, over $\text{Gest}(D_{\geq 0}(k))$ or $\text{Gest}(D(k))$, finite sets are 0-affine, 0-proper, and 0-étale.

To see that finite sets are always 1-affine, note that by shifting this is equivalent to finite sets being 0-affine when working over $X = \text{Gest}(1\text{Pr})$; and 1Pr is semiadditive.

8.4. Countable sets. Generalizing the previous example to countable sets, they are still 1-affine and hence 1-proper, and 0-étale. Indeed, as a countable colimit of 0-étale maps, they are certainly 0-étale; this already proves that they are 1-proper. To prove 1-affineness, we argue as in the previous part, but noting that after shifting 1Pr is even countably semiadditive. (More generally, countable sets become 0-affine after base change to any $X = \text{Gest}(A)$ with A_1 countably semiadditive.)

8.5. Countable anima. Again, countable anima are 0-étale and hence 1-proper. In general, they are however not n -affine for any n , even after base change to say $\text{Gest}(D(k))$ for reasonable k . Indeed, we prove by induction on n that $B^{n+1}(\mathbb{Z}/2)$ is not n -affine, when working over $\text{Gest}(D(\mathbb{Q}))$ (and hence their disjoint union is not n -affine for any n). For this, we first note that $B^{n+1}(\mathbb{Z}/2)$ is in fact 0-proper over $\text{Gest}(D(\mathbb{Q}))$, by induction on n . By induction, and as we already know 1-properness, it suffices to see that it is 0-prim. This means that the functor

$$D(\mathbb{Q}) \rightarrow D(\mathbb{Q})^{B^{n+1}(\mathbb{Z}/2)}$$

has a $D(\mathbb{Q})$ -linear right adjoint, i.e. preserves small colimits. For $n = 0$, this is the functor from $D(\mathbb{Q})$ to $\mathbb{Z}/2$ -representations in $D(\mathbb{Q})$, and the right adjoint takes $\mathbb{Z}/2$ -cohomology. As 2 is invertible in $\mathbb{Q}$, $\mathbb{Z}/2$ -cohomology is exact and hence commutes with all small colimits. For $n > 0$, the pullback functor is actually an equivalence. Thus, by Lemma 6.15, if $B^{n+1}(\mathbb{Z}/2)$ is n -affine for $n \geq 1$, then its diagonal is $(n-1)$ -affine, and hence by base change $B^n(\mathbb{Z}/2)$ is $(n-1)$ -affine. Thus, to prove that $B^{n+1}(\mathbb{Z}/2)$ is not n -affine, it suffices to see that $B(\mathbb{Z}/2)$ is not 0-affine. But its 0-affinization corresponds to the algebra of $\mathbb{Z}/2$ -cohomology on $\mathbb{Q}$, which is just $\mathbb{Q}$.

Of course, this argument works more generally, for $\mathbb{Z}/2$ replaced by any finite abelian group, and $\mathbb{Q}$ by any ring in which its order is invertible. Even more generally, we have the following result, showing that in large generality passing to the diagonal drops the level of affineness by precisely one. Part (2) generalizes results of Gaitsgory on 1-affineness [Gai15], and related results for the general case appeared in Stefanich's thesis [Ste21].

PROPOSITION 8.1. *Let $f : Y \rightarrow X$ be a map of Gestalten and $n \geq 0$.*

- (1) *Assume that f is $n+1$ -affine and Δ_f is n -prim. Then Δ_f is n -affine.*
- (2) *Assume that Δ_f is n -affine and f is $n+1$ -prim. Then f is $n+1$ -affine.*

In particular, if f is n -proper, then f is $n+1$ -affine if and only if Δ_f is n -affine.

PROOF. Part (1) follows from Lemma 6.15 applied to $\Delta_f^* : A = \mathcal{O}(Y \times_X Y) \rightarrow B = \mathcal{O}(Y)$. For part (2), let $Y' = \text{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_{n+1})$ be the $n+1$ -affine truncation of Y/X . By Corollary 8.14, the map $h : Y \rightarrow Y'$ is surjective. Thus, it suffices to see that $\Delta_h : Y \rightarrow Y \times_{Y'} Y$ is an isomorphism. We first note that Δ_h is n -affine. Indeed, $\Delta_f : Y \rightarrow Y \times_X Y$ is n -affine, so it suffices to see that $Y \times_{Y'} Y \rightarrow Y \times_X Y$ is also n -affine. This is a pullback of the diagonal $Y' \rightarrow Y' \times_X Y'$, which arises from the n -affine map Δ_f by taking $n+1$ -affine truncations over X , and hence is also n -affine. But $\Delta_h : Y \rightarrow Y \times_{Y'} Y$ also induces an isomorphism upon taking $(\mathcal{O}(-)/\mathcal{O}(Y'))_{n+1}$ as $h : Y \rightarrow Y'$ is $n+1$ -prim so that

$$(\mathcal{O}(Y \times_{Y'} Y)/\mathcal{O}(Y'))_{n+1} = (\mathcal{O}(Y)/\mathcal{O}(Y'))_{n+1} \otimes_{\mathcal{O}(Y')_{n+2}} (\mathcal{O}(Y)/\mathcal{O}(Y'))_{n+1},$$

but $(\mathcal{O}(Y)/\mathcal{O}(Y'))_{n+1} = 1$ in $\mathcal{O}(Y')_{n+2}$. Thus, $\Delta_h : Y \rightarrow Y \times_{Y'} Y$ is n -affine but also induces an isomorphism $\Delta_{h,n}^*$ on $\mathcal{O}(-)_{n+1}$; this implies that it is an isomorphism (as the map $1 \rightarrow \Delta_{h,n}^* 1$ in $\mathcal{O}(Y \times_{Y'} Y)_{n+1}$ is an isomorphism, by virtue of $\Delta_{h,n}^*$ being an equivalence). $\square$

8.6. Affine line. Consider the functor taking any $Y = \text{Gest}(B) \in \text{Gest}$ to $\mathcal{O}(Y)_0 = B_0 \in \text{Ani}$. Working over a base $X = \text{Gest}(A)$, this is representable by $\mathbb{A}^1 := \text{Gest}_X(A_0\{x\})$, where $A_0\{x\}$ is the free E_∞ -algebra on one generator x over A_0 . In particular, it is 0-affine, and hence 0-proper and 1-étale. In particular, over the absolute base, this is $\text{Gest}(*\{x\})$ where $*\{x\} = \bigsqcup_{n \geq 0} */\Sigma_n$ is the free E_∞ -monoid (in anima) on one generator.

Note that over $\mathbb{S}$ or even $\mathbb{Z}$, the free E_∞ -algebra $\mathbb{Z}\{x\} = \bigoplus_{n \geq 0} \mathbb{Z}[*/\Sigma_n]$ is not isomorphic to the polynomial algebra $\mathbb{Z}[x]$; rather, it has contributions coming from the group homology of symmetric groups. After base change to $\mathbb{Q}$, the two versions of the polynomial algebra agree.

We can also define the other version of the affine line already over the absolute base; we will denote it by $\mathbb{G}_a$. This is also 0-affine, and over the absolute base it is $\mathbb{G}_a = \text{Gest}(\mathbb{N})$, or over any base $X = \text{Gest}(A)$ as $\text{Gest}_X(A_0[x])$ where $A_0[x] = \bigsqcup_{n \geq 0} A_0 \cdot x^n$ denotes the polynomial algebra over A_0 . Again, it is 0-affine and hence 0-proper and 1-étale.

8.7. Multiplicative group. Consider the subfunctor $\mathrm{GL}_1 \subset \mathbb{A}^1$ taking $Y = \mathrm{Gest}(B)$ to the invertible elements $\mathcal{O}(Y)_0^\times = B_0^\times \subset \mathcal{O}(Y)_0 = B_0$. This is again representable and 0-affine (hence 0-proper, 1-étale), and given by $\mathrm{Gest}_X(A_0\{x^{\pm 1}\})$, the free E_∞ -algebra on an invertible element x . Again, there is also another version, defined as

$$\mathbb{G}_m := \mathrm{GL}_1 \times_{\mathbb{A}^1} \mathbb{G}_a = \mathrm{Gest}_X(A_0[x^{\pm 1}]).$$

Over the absolute base, this is $\mathrm{Gest}(\mathbb{Z})$ (where $\mathbb{Z} \in \mathrm{CAlg}(\mathrm{Ani})$ as a “multiplicative” group). Thus, it represents the functor taking any $Y = \mathrm{Gest}(B)$ to $\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Ani})}(\mathbb{Z}, B_0)$. Again, over $D(\mathbb{Q})$ (or just $D_{\geq 0}(\mathbb{Q})$), GL_1 and $\mathbb{G}_m$ are the same.

8.8. Projective line. As an example of a gluing that one can do in Gestalten, we can form the pushout

$$\mathbb{A}^1 \sqcup_{\mathrm{GL}_1} \mathbb{A}^1$$

(where the maps $\mathrm{GL}_1 \rightarrow \mathbb{A}^1$ are $x \mapsto x$ and $x \mapsto x^{-1}$). This is already defined over the absolute base. As a pushout of 1-étale Gestalten, it is 1-étale. It is not 0-affine, but its diagonal is 0-affine. Indeed, the product of two copies of $\mathbb{A}^1 \sqcup_{\mathrm{GL}_1} \mathbb{A}^1$ is covered by four copies of $\mathbb{A}^1 \times \mathbb{A}^1$, and the preimage of the diagonal is either $\mathbb{A}^1$ embedded diagonally, or GL_1 embedded via $x \mapsto (x, x^{-1})$; both are 0-affine as maps between 0-affines.

It turns out that before passing to the stable setting of $D(\mathbb{S})$, this is not well-behaved, and different from the object one might expect. Namely, Zariski covers are not usually covers in $D_{\geq 0}(k)$:

PROPOSITION 8.2. *Let k be a field, for concreteness. The map*

$$\mathrm{Gest}_X(k[T^{\pm 1}]) \sqcup \mathrm{Gest}_X(k[T, \frac{1}{1-T}]) \rightarrow \mathrm{Gest}_X(k[T])$$

over $X = \mathrm{Gest}(D_{\geq 0}(k))$ is not surjective. After base change to $\mathrm{Gest}(D(k))$, it becomes surjective.

PROOF. After base change to $D(k)$, this follows from Theorem 4.8 and Theorem 7.7. Before, however, one would need that the image of the map

$$D_{\geq 0}(k[T^{\pm 1}]) \times D_{\geq 0}(k[T, \frac{1}{1-T}]) \rightarrow D_{\geq 0}(k[T])$$

generates the target under small colimits, by Proposition 8.11 (ii). But $k[T]$ does not admit maps from nonzero $k[T^{\pm 1}]$ or $k[T, \frac{1}{1-T}]$ -modules, so it cannot be generated in this way. $\square$

After base change to $D(\mathbb{S})$, we see from Theorem 4.8 and Theorem 7.7 that

$$\mathbb{A}^1 \sqcup_{\mathrm{GL}_1} \mathbb{A}^1$$

is coming from the obvious fpqc stack (which here is really just a derived scheme). In particular, it is 0-proper. If one forms the version

$$\mathbb{G}_a \sqcup_{\mathbb{G}_m} \mathbb{G}_a$$

using actual polynomial algebras instead of free E_∞ -rings, then it is also 0-suave. Over $D(\mathbb{Q})$, the difference disappears.

In the following examples, we work over $X = D(\mathbb{Q})$.

8.9. The classifying stack of $\mathbb{G}_m$. Related to the projective line, we also have the classifying stack $*/\mathbb{G}_m$. A related object is the moduli space Pic of line bundles, taking any $Y = \text{Gest}(B)$ to the anima $B_1^\times$ of invertible objects in $\mathcal{O}(Y)_1 = B_1$. Note that for any line bundle $L \in B_1^\times$, one has the commutativity isomorphism $c : L \otimes L \cong L \otimes L$ which must square to the identity; this yields a map $\text{Pic} \rightarrow \mu_2$ where $\mu_2 = \text{Gest}_X(\mathbb{Q}[x]/(x^2 - 1))$ is the group of second roots of unity. In fact, this is isomorphic to $\mathbb{Z}/2$ (as we inverted 2).

PROPOSITION 8.3. *The natural maps*

$$*/\mathbb{G}_m \rightarrow \text{Pic} \rightarrow \mu_2$$

yield a fibre sequence of sheaves of spectra on Gestalten over $\text{Gest}(D(\mathbb{Q}))$.

PROOF. The example of the invertible object $\mathbb{Q}[1] \in D(\mathbb{Q})$ shows that the map $\text{Pic} \rightarrow \mu_2$ is surjective. We need to see that any $L \in A_1^\times$ for which the commutativity isomorphism $c : L \otimes L \cong L \otimes L$ is the identity is locally trivial. Note that in this case the action of Σ_n on $L^{\otimes n}$ is (uniquely) trivial, for all $n \geq 1$. Indeed, this action is given by a map $\Sigma_n \rightarrow \text{Aut}(L^{\otimes n}) = \text{Aut}(1) = A_0^\times$. But $A_0^\times$ is a connective $\mathbb{E}_\infty$ -group in anima all of whose π_i for $i > 0$ are $\mathbb{Q}$ -vector spaces (as A_0 refines to a commutative algebra in $D_{\geq 0}(\mathbb{Q})$). It then follows that as the map $\Sigma_n \rightarrow \pi_0(A_0^\times)$ is trivial (as it is trivial on transpositions by assumption), the map $\Sigma_n \rightarrow A_0^\times$ is (uniquely) trivial.

It follows that we can form the graded algebra

$$B = \bigoplus_{n \in \mathbb{Z}} L^{\otimes n} \in \text{CAlg}(A_1)$$

Then $L \otimes_{A_1} B \cong B$ in $\text{Mod}_B(A_1)$, so the line bundle becomes trivial after base change along $\text{Gest}(B) \rightarrow \text{Gest}(A)$. (Here $\text{Gest}(B)$ denotes the 0-affine Gestalt over $\text{Gest}(A)$ associated to B .) Moreover, $\text{Gest}(B) \rightarrow \text{Gest}(A)$ is a cover as the map $1 \rightarrow B$ splits in A_1 , cf. Proposition 8.11 (i). This shows that the map

$$* \rightarrow \text{fib}(\text{Pic} \rightarrow \mu_2)$$

is surjective.

On the other hand, it is clear that $* \times_{\text{Pic}} *$ classifies automorphisms of the trivial line bundle, which is $\mathbb{G}_m$. Thus, the image of $* \rightarrow \text{fib}(\text{Pic} \rightarrow \mu_2)$ is $*/\mathbb{G}_m$. $\square$

We note that $\pi_0 \text{Pic}(D(\mathbb{Q})) = \mathbb{Z}$, given by shifts $\mathbb{Q}[n]$ for various $n \in \mathbb{Z}$. However, the proof shows that for even n , these classes are locally trivial in Gestalten. To trivialize them, one has to pass to the periodic E_∞ -rings $\bigoplus_{m \in \mathbb{Z}} \mathbb{Q}[nm]$. Only if n is even this has a commutative multiplication.

We will later study higher versions of this story, comparing $B^n \mathbb{G}_m$ and the moduli space of invertible objects of A_n . In the terminology used in the literature on this subject, this studies “invertible topological quantum field theories”; these are expected to be classified by a spectrum that is essentially an Anderson dual of the sphere. In this optic, the μ_2 above should be seen as the dual of the first stable homotopy group of spheres $\pi_1 \mathbb{S} = \mathbb{Z}/2$. We will in fact see that this moduli space of invertible objects in the A_n defines a sheaf of spectra on Gestalten that is an internal Anderson dual of the sphere, proving a precise version of this expectation.

8.10. De Rham stacks. Let k be a field of characteristic 0 and let X be an algebraic variety over k , considered as a Gestalt over $\text{Gest}(D(k))$. Recall that the de Rham stack X_{dR} is the stackification of the functor taking a k -algebra R to $X(R_{\text{red}})$.

PROPOSITION 8.4. *The map $X \rightarrow X_{\text{dR}}$ identifies X_{dR} as the 0-truncation $X_{\text{dR}} = \pi_0 X$ in Gestalten.*

PROOF. Recall that any nilpotent immersion $\text{Spec}(R) \rightarrow \text{Spec}(S)$ yields a descendable map, cf. [Mat16, Proposition 3.33]. This implies that $X \rightarrow X_{\text{dR}}$ is surjective, as for any map $\text{Spec}(S) \rightarrow X_{\text{dR}}$ one can find a nilpotent immersion $\text{Spec}(R) \rightarrow \text{Spec}(S)$ which lifts to X . By a similar reasoning, the map $X \rightarrow (X \subset X \times X)^\wedge$ from X to the formal completion of the diagonal is surjective. As $(X \subset X \times X)^\wedge$ is a sub-Gestalt of $X \times X$, this shows that $\pi_0 X$ is the quotient of X by $(X \subset X \times X)^\wedge \subset X \times X$, which is also X_{dR} . $\square$

A related fact is the following. On Gestalten over $D(\mathbb{Q})$, the functor $X \mapsto (\mathcal{O}(X)/\mathbb{Q})_0 \in D(\mathbb{Q})$ defines a sheaf $\mathcal{O}_0$ with values in $\text{CAlg}(D(\mathbb{Q}))$.

PROPOSITION 8.5. *The sheaf $\mathcal{O}_0$ on $\text{Gest}_{/D(\mathbb{Q})}$ is weakly even periodic, i.e. $\pi_{2i-1}\mathcal{O}_0$ vanishes, $\pi_{2i}\mathcal{O}_0$ is an invertible module over $\pi_0\mathcal{O}_0 = \mathbb{A}_{\text{dR}}^1$, and $\pi_{2i}\mathcal{O}_0 \cong (\pi_2\mathcal{O}_0)^{\otimes i}$ as invertible $\pi_0\mathcal{O}_0 = \mathbb{A}_{\text{dR}}^1$ -modules.*

PROOF. Covering $\text{Gest}(D(\mathbb{Q}))$ by $\text{Gest}(D(\bigoplus_{n \in \mathbb{Z}} \mathbb{Q}[2n]))$, we see that $\mathcal{O}_0$ becomes locally two-periodic. It remains to see that $\pi_1\mathcal{O}_0$ is zero. But this follows from $\mathbb{Q}[x] \rightarrow \mathbb{Q}$ being descendable, when x is a generator of homological degree 1 (which hence squares to zero). $\square$

Appendix to Lecture VIII: Descendability

The goal of this appendix is to understand what it means for a map $f : Y = \text{Gest}(B) \rightarrow X = \text{Gest}(A)$ to be surjective, and relate it to Mathew's notion of descendability [Mat16] in practical situations.

PROPOSITION 8.6. *Let $f : Y \rightarrow X$ be an n -suave map of Gestalten. Then the image of f is an n -suave injection $j : U \subset X$.*

PROOF. The image of f is the colimit of the Čech nerve of $Y \rightarrow X$, all of whose terms are n -suave over X , hence so is the colimit. $\square$

PROPOSITION 8.7. *Let $f : Y \rightarrow X$ be an injective map of Gestalten, i.e. the diagonal of f is an isomorphism $Y \times_X Y = Y$.*

- (i) *The map f is n -étale if and only if it is n -suave. In this case, $(\mathcal{O}(Y)/\mathcal{O}(X))_n^! \in \mathcal{O}(X)_{n+1}$ is a coidempotent coalgebra, and this induces an equivalence between n -étale injections into X , and coidempotent coalgebras in $\mathcal{O}(X)_{n+1}$.*
- (ii) *The map f is n -affine if and only if it is n -proper if and only if it is n -prim. In this case, $(\mathcal{O}(Y)/\mathcal{O}(X))_n \in \mathcal{O}(X)_{n+1}$ is an idempotent algebra, and this induces an equivalence between n -proper injections into X , and idempotent algebras in $\mathcal{O}(X)_{n+1}$.*

PROOF. As the diagonal of f is an isomorphism, it is clear that n -étale is equivalent to n -suave. If $j : U \rightarrow X$ is n -étale injection, then $(\mathcal{O}(U)/\mathcal{O}(X))_n^! \in \mathcal{O}(X)_{n+1}$ is coidempotent, as its tensor square in general yields $(\mathcal{O}(U \times_X U)/\mathcal{O}(X))_n^!$. Conversely, given a coidempotent coalgebra $C \in \mathcal{O}(X)_{n+1}$, the comodules over C define a full subcategory of $\mathcal{O}(X)_{n+1}$ for which $- \otimes C$ provides a symmetric monoidal right adjoint functor; passing to the corresponding Gestalt yields an n -étale injection giving rise to the given C . As both n -étale injections and coidempotent coalgebras define partially ordered sets, it remains to see that if $(\mathcal{O}(U)/\mathcal{O}(X))_n^! = 1 \in \mathcal{O}(X)_{n+1}$, then $U = X$. To see this, it is inductively enough to show that $(\mathcal{O}(U)/\mathcal{O}(X))_{n+1}^! = 1 \in \mathcal{O}(X)_{n+2}$. But the map $(\mathcal{O}(U)/\mathcal{O}(X))_{n+1}^! \rightarrow 1$ has a right adjoint such that the counit of the adjunction is the map $(\mathcal{O}(U)/\mathcal{O}(X))_n^! \rightarrow 1$ in $\mathcal{O}(X)_{n+1} = \text{End}_{\mathcal{O}(X)_{n+2}}(1)$; this implies that $(\mathcal{O}(U)/\mathcal{O}(X))_{n+1}^! \rightarrow 1$ is a localization. As $(\mathcal{O}(U)/\mathcal{O}(X))_{n+1}^! \rightarrow 1$ is also coidempotent, this implies that it must be an isomorphism.

The proof of part (ii) is similar, but slightly easier (as the inverse operation is just given by passing to the Gestalt of the idempotent algebra). $\square$

Using this, we can also describe complements.

PROPOSITION 8.8. *Let $f : Y \rightarrow X$ be an n -étale injection of Gestalten, corresponding to a coidempotent coalgebra $C = (\mathcal{O}(Y)/\mathcal{O}(X))_n^! \in \mathcal{O}(X)_{n+1}$. Let $A = \text{cofib}(C \rightarrow 1) \in \mathcal{O}(X)_{n+1}$ be the associated idempotent algebra. Then the complement of Y in X — i.e., the subfunctor of X of all maps $S \rightarrow X$ such that $S \times_X Y = \emptyset$ — is the n -proper injection $g : Z = \text{Gest}(A) \rightarrow X$ determined by A .*

If f is instead n -proper, then to describe the complement, one has to instead consider it as $n+1$ -étale, and apply the proposition one level higher.

PROOF. First recall that if C is any coidempotent coalgebra, then $A = \text{cofib}(C \rightarrow 1)$ is an idempotent algebra, as one computes $A \otimes A = \text{cofib}(A \otimes C \rightarrow A) = A$, noting that $A \otimes C =$

$\mathrm{cofib}(C \otimes C \rightarrow C) = 0$. Unless we are in a stable situation — which can only happen for $n = 0$ — we can however not go back, and it may happen that $A = 0$ even if $C \rightarrow 1$ is not an isomorphism.

In any case, a map $S \rightarrow X$ satisfies $S \times_X Y$ if and only if the base change of C to S vanishes. This implies that S factors over $\mathrm{Gest}(A)$, as the map $1 \rightarrow A = \mathrm{cofib}(C \rightarrow 1)$ becomes an isomorphism after base change to S . Conversely, we saw above that after base change to A , one has $A \otimes C = 0$. $\square$

DEFINITION 8.9.

- (i) An open immersion of Gestalten is a 0-étale injection.
- (ii) A closed immersion of Gestalten is a 0-proper injection.

By Proposition 8.8, the complement of an open immersion always yields a closed immersion; the converse is true when we work over $D(\mathbb{S})$, i.e. in the stable setting.

Now we can characterize surjectivity as follows; this is modelled on [Sch25b, Proposition 6.19].

PROPOSITION 8.10. *Let $f : Y \rightarrow X$ be an n -suave map of Gestalten. The following conditions are equivalent.*

- (i) *The map f is surjective.*
- (ii) *The map*

$$\mathrm{colim}_{[m] \in \Delta^{\mathrm{op}}} (\mathcal{O}(Y^{\times_X (m+1)}) / \mathcal{O}(X))_n^! \rightarrow 1$$

is an isomorphism in $\mathcal{O}(X)_{n+1}$.

- (iii) *Denoting the coalgebra $C = (\mathcal{O}(Y) / \mathcal{O}(X))_n^!$ in $\mathcal{O}(X)_{n+1}$, the map*

$$\mathrm{colim}_{[m] \in \Delta^{\mathrm{op}}} C^{\otimes m+1} \rightarrow 1$$

is an isomorphism in $\mathcal{O}(X)_{n+1}$.

- (iv) *The pullback functor*

$$\mathcal{O}(X)_{n+1} \rightarrow \mathcal{O}(Y)_{n+1}$$

is conservative.

PROOF. Parts (ii) and (iii) are equivalent by the Künneth formula. Parts (i) and (ii) are equivalent by Proposition 8.6 and Proposition 8.7 (i). Being a cover certainly implies that pullback is conservative. Conversely, if pullback $\mathcal{O}(X)_{n+1} \rightarrow \mathcal{O}(Y)_{n+1}$ is conservative, one can check that the map in (iii) is an isomorphism after pullback to Y . But then this computes the similar colimit for $Y \times_X Y \rightarrow Y$, which is clearly a cover as it splits. $\square$

If $n = 0$ this is easy to check in practice; for higher values of n it becomes more difficult. For $n \geq 1$, one can still say a bit more. Roughly, being a cover is closely related to the functor

$$f_{n-1,!} : \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y) / \mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n$$

generating the target “quickly”. We note that if f is $(n-1)$ -proper, then $\mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y) / \mathcal{O}(X))_n^!)$ can be identified with $\mathcal{O}(Y)_n$, and the displayed functor is identified with

$$f_{n-1,*} : \mathcal{O}(Y)_n \rightarrow \mathcal{O}(X)_n.$$

PROPOSITION 8.11. *Let $f : Y \rightarrow X$ be an n -suave map of Gestalten with $n \geq 1$.*

- (i) *Consider the map*

$$\mathrm{colim}_{[m] \in \Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y^{\times_X (m+1)}) / \mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n$$

of presentable $(\infty, 1)$ -categories. If $1 \in \mathcal{O}(X)_n$ is a retract of an object in the image of this functor, then f is surjective. In particular, if $1 \in \mathcal{O}(X)_n$ is a retract of an object in the image of

$$f_{n-1,!} : \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y)/\mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n,$$

then f is surjective.

- (ii) Conversely, assume that X is n -prim, or just that $n\mathrm{Pr} \rightarrow \mathcal{O}(X)_{n+1}$ has a colimit-preserving right adjoint. Then if f is surjective, the map

$$\mathrm{colim}_{[m] \in \Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y^{\times_X(m+1)})/\mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n$$

is an equivalence of presentable $(\infty, 1)$ -categories, and the image of

$$f_{n-1,!} : \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y)/\mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n$$

generates the target under small colimits; more precisely, any object in $\mathcal{O}(X)_n$ is a geometric realization of a simplicial object taking values in objects in the image of $f_{n-1,!}$.

QUESTION 8.12. In part (i), can the condition of being a retract of an object in the image be weakened to the condition of being a geometric realization (or a sifted colimit) of objects in the image? We note that one cannot allow arbitrary colimits, as the example of X with $\mathcal{O}(X)_1$ stable, and $Y = U \sqcup Z$ for an open immersion $j : U \subset X$ with closed complement $i : Z \subset X$ shows — this is not a cover, yet $1 \in \mathcal{O}(X)_1$ is a colimit of $j_!1$ and i_*1 .

PROOF. Let us first prove (ii) which is the easy part. Namely, the colimit-preserving right adjoint is $\mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, -)$, so the first assertion comes from Proposition 8.10. Then any $M \in \mathcal{O}(X)_n$ can be written as the geometric realization of the composites

$$\mathcal{O}(X)_n \rightarrow \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, (\mathcal{O}(Y^{\times_X(m+1)})/\mathcal{O}(X))_n^!) \rightarrow \mathcal{O}(X)_n$$

where the first map is the right adjoint to the second map (which exists on the level of $(\infty, 1)$ -categories). This is a geometric realization of a simplicial object taking values in objects in the image of $f_{n-1,!}$.

For part (i), assume that $1 \in \mathcal{O}(X)_n$ is a retract of an object in the image. Letting

$$C = \mathrm{colim}_{[m] \in \Delta^{\mathrm{op}}} (\mathcal{O}(Y^{\times_X(m+1)})/\mathcal{O}(X))_n^! \in \mathcal{O}(X)_{n+1}$$

be the coidempotent coalgebra describing the image of f , we see in particular that $1 \in \mathcal{O}(X)_n$ is a retract of an object in the image of

$$\mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(C, 1) \rightarrow \mathrm{Hom}_{\mathcal{O}(X)_{n+1}}(1, 1) = \mathcal{O}(X)_n.$$

In particular, there is a map $G : 1 \rightarrow C$ such that, denoting $F : C \rightarrow 1$ the counit of C , the identity endomorphism of $1 \in \mathcal{O}(X)_{n+1}$ is a retract of GF . By Lemma 8.13 below applied to $\mathbb{C} = \mathcal{O}(X)_{n+1}$, this implies that $1 \in \mathcal{O}(X)_{n+1}$ lies in the $1\mathrm{Pr}$ -linear subcategory of C -comodules, hence the category of C -comodules is all of $\mathcal{O}(X)_{n+1}$, which means that f is a cover. $\square$

For the proof, we used the following lemma about 2-retracts. This is a special case of a notion of higher retracts, termed “condensations” by Gaiotto–Johnson–Freyd [GJF25].²³

²³The reader is extremely encouraged to open their paper in order to appreciate a wildly different perspective on the kind of higher-categorical concepts that are also studied in these lectures.

LEMMA 8.13. *Let $\mathbb{C}$ be a presentable $(\infty, 2)$ -category. Let $X, Y \in \mathbb{C}$ and assume that there are maps $F : Y \rightarrow X$, $G : X \rightarrow Y$ such that id_Y is a retract of GF in $\text{End}_{\mathbb{C}}(Y)$. Then Y lies in the 1Pr-linear subcategory of $\mathbb{C}$ generated by X .*

PROOF. We can replace $\mathbb{C}$ by the 1Pr-linear category that is atomically generated by X and Y . Let $i : \mathbb{C}_X \subset \mathbb{C}$ be the subcategory generated by X . The inclusion i admits a 1Pr-linear right adjoint i^R ; let $Y' = i^R(Y)$. Then by applying i^R , the object Y' carries all the structure that Y carries – there are maps $F' : Y' \rightarrow X$, $G' : X \rightarrow Y'$ so that $\text{id}_{Y'}$ is a retract of $G'F'$. Moreover, there is a comparison map $H : Y' \rightarrow X$ (where we abusively denote by Y' also its image $i(Y') \in \mathbb{C}$) that is compatible with this extra structure. Our goal is to show that H is an isomorphism.

By Yoneda, we can now reduce to $\mathbb{C} = 1\text{Pr}$. We need to see that $H : Y' \rightarrow Y$ is an equivalence. As any object of Y is a retract of an object in the image of H (as id_Y is a retract of GF , and $G = HG'$ factors over Y'), it suffices to prove fully faithfulness. Moreover, it is enough to prove fully faithfulness on the image of $G' : X \rightarrow Y'$. Thus, let $x_1, x_2 \in X$. We want to show that the map

$$\text{Hom}_{Y'}(G'(x_1), G'(x_2)) \rightarrow \text{Hom}_Y(G(x_1), G(x_2))$$

is an isomorphism. But $\text{Hom}_Y(G(x_1), G(x_2))$ is a retract of $\text{Hom}_X(FG(x_1), FG(x_2))$ — indeed, by applying F there is a map from the source to the target, and by applying G we can go back to $\text{Hom}_Y(GFG(x_1), GFG(x_2))$, from where we can project back to $\text{Hom}_Y(G(x_1), G(x_2))$ by using that $G(x_i)$ is a retract of $GFG(x_i)$. This description also applies analogously to $\text{Hom}_{Y'}(G'(x_1), G'(x_2))$, writing it as a retract of $\text{Hom}_X(F'G'(x_1), F'G'(x_2))$. But $F'G' = FG$, so this shows that $\text{Hom}_{Y'}(G'(x_1), G'(x_2))$ and $\text{Hom}_Y(G(x_1), G(x_2))$ are both given by the same retract of $\text{Hom}_X(FG(x_1), FG(x_2))$. $\square$

As one application, we have the following general surjectivity result for prim maps.

COROLLARY 8.14. *Let $f : Y \rightarrow X$ be a map of Gestalten that is n -prim, and eventually suave, i.e. m -suave for some sufficiently large m . Then the map*

$$Y \rightarrow \text{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_n)$$

from Y to its n -affine truncation is surjective.

In particular, f is surjective if and only if the n -affine map

$$\text{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_n) \rightarrow X$$

is surjective.

PROOF. Assume first that we can take $m = n+2$. By shifting, we can assume $n = 0$. We can also replace X by $\text{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_n)$ and assume $(\mathcal{O}(X)/\mathcal{O}(Y))_n = 1$. After these replacements $f : Y \rightarrow X$ is a 0-prim and 2-suave map, and $f_{0*}1 = 1$, and we want to show that f is surjective. Note that as f is both 2-prim and 2-suave, $(\mathcal{O}(Y)/\mathcal{O}(X))_2^! \in \mathcal{O}(X)_3$ is dualizable (by Lemma 6.22) with dual $(\mathcal{O}(Y)/\mathcal{O}(X))_2$. Moreover, as f is 1-prim, the unit map $1 \rightarrow (\mathcal{O}(Y)/\mathcal{O}(X))_2$ has a right adjoint in $\mathcal{O}(X)_3$, and hence the counit map $(\mathcal{O}(Y)/\mathcal{O}(X))_2^! \xrightarrow{f_{1,!}} 1$ has a left adjoint in $\mathcal{O}(X)_3$. Moreover, the composite

$$1 \rightarrow (\mathcal{O}(Y)/\mathcal{O}(X))_2^! \xrightarrow{f_{1,!}} 1$$

agrees with the composite

$$1 \xrightarrow{f_1^*} (\mathcal{O}(Y)/\mathcal{O}(X))_2 \xrightarrow{f_{1,*}} 1$$

obtained by dualizing, and this is given by tensoring with $f_{1*}1$. But $f_{1*}1 \in \mathcal{O}(X)_2$ admits 1 as a retract: Indeed, we have the unit map $1 \rightarrow f_{1*}1$, which has a right adjoint, and the composite map $1 \rightarrow f_{1*}1 \rightarrow 1$ is given by tensoring with $f_{0*}1 = 1$. It follows that $1 \in \mathcal{O}(X)_2$ is a retract of an object in the image of

$$f_{1,!} : \mathrm{Hom}_{\mathcal{O}(X)_3}(1, (\mathcal{O}(Y)/\mathcal{O}(X))_2^!) \rightarrow \mathcal{O}(X)_2,$$

so Proposition 8.11 (i) shows that f is surjective.

For general m (without loss of generality $m > n + 1$), let $Y_k = \mathrm{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_k)$ be the k -affine truncation for $k = n, \dots, m - 2$. We claim that all maps

$$Y \rightarrow Y_{m-2} \rightarrow \dots \rightarrow Y_{k+1} \rightarrow Y_k$$

are surjective. For the first map $Y \rightarrow Y_{m-2}$, this follows from the case already established. But now $g : Y_{m-2} \rightarrow X$ is $m - 2$ -affine and n -prim (as for $k \leq m - 2$ the map $1 \rightarrow g_{k*}1$ agrees with the map $1 \rightarrow f_{k*}1$, while for $k > m - 2$ it has a right adjoint as g is $m - 2$ -affine), and hence $m - 1$ -suave. Thus, we can apply the case already established to show that $Y_{m-2} \rightarrow Y_{m-3}$ is surjective, and induct. $\square$

Now we can also relate surjectivity to the notion of descendability of Mathew [Mat16]. This corollary is closely related to [Sch25b, Proposition 6.20].

COROLLARY 8.15. *Let $f : Y \rightarrow X$ be a 0-prim and eventually suave map of Gestalten. Assume that $\mathcal{O}(X)_1$ is stable, and $f_{0*}(1) \in \mathcal{O}(X)_1$ is descendable. Then f is surjective.*

PROOF. By Corollary 8.14, we can assume that Y is 0-affine. In particular, it is 0-proper, and we can identify $(\mathcal{O}(Y)/\mathcal{O}(X))_1^! = (\mathcal{O}(Y)/\mathcal{O}(X))_1$, and similarly for finite products of Y over X . Thus, for any finite m , the functor

$$\mathrm{colim}_{\Delta_{\leq m}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{O}(X)_2}(1, (\mathcal{O}(Y^{\times_X (\bullet+1)})/\mathcal{O}(X))_1^!) = \mathrm{colim}_{\Delta_{\leq m}^{\mathrm{op}}} (\mathcal{O}(Y^{\times_X \bullet+1})/\mathcal{O}(X))_1 \rightarrow \mathcal{O}(X)_1$$

has a left adjoint; the unit of the adjunction is the finite limit of the units of the adjunctions at the individual levels. (We use here that finite limits are also finite colimits in stable $(\infty, 1)$ -categories.) Thus, evaluated at $1 \in \mathcal{O}(X)_1$, this unit is the map

$$1 \rightarrow \lim_{\Delta_{\leq m}^{\mathrm{op}}} f_{0*}(1)^{\otimes \bullet+1}.$$

By descendability of $f_{0*}(1)$, there is some m for which this map admits a retract. Now Proposition 8.11 (i) ensures that f is surjective. $\square$

In particular, in the following situation, we get a clean statement.

COROLLARY 8.16. *Let X be a 0-prim Gestalt with $\mathcal{O}(X)_1$ stable. Let $f : Y \rightarrow X$ be 0-prim and eventually suave. Then f is surjective if and only if $f_{0*}(1) \in \mathcal{O}(X)_1$ is descendable.*

PROOF. By Corollary 8.14, we can assume that f is 0-affine. In one direction, this follows from Corollary 8.15. In the converse direction, this follows from Proposition 8.11 (ii). Indeed, the image of f_{0*} is contained in $f_{0*}(1)$ -modules, and so if X is 0-prim and hence $1 \in \mathcal{O}(X)_1$ is compact, then the unit is a finite colimit of $f_{0*}(1)$ -modules, which implies that $f_{0*}(1)$ is descendable. $\square$

9. Lecture IX: Six Functors and Gestalten

How does one construct interesting examples of Gestalten? In the examples we saw so far we mostly started from commutative algebras (in simple symmetric monoidal base categories, like $D(\mathbb{S})$ or $D(k)$), defining 0-affine Gestalten, and then built more general ones by gluing, i.e. taking colimits.

As a particular example, note that this can be used to define a functor from light condensed anima to Gestalten over $D(\mathbb{S})$. Namely, any light profinite set $S = \lim_i S_i$ can be sent to

$$\mathrm{Gest}_{\mathrm{Gest}(D(\mathbb{S}))}(\mathrm{Cont}(S, \mathbb{S})) = \lim_i (S_i \times \mathrm{Gest}(D(\mathbb{S}))).$$

(This could of course already be defined over the absolute base $\mathrm{Gest}(\mathrm{Ani})$. For the following proposition, it is important to work stably.) By Theorem 4.8, this takes hypercovers to colimits in Gestalten, so we see that the functor

$$\mathrm{ProFin}^{\mathrm{light}} \rightarrow \mathrm{Gest}_{\mathrm{Gest}(D(\mathbb{S}))}^{0\text{-aff}}$$

extends to a countable colimit and finite-limit-preserving functor

$$\mathrm{CondAni}^{\mathrm{light}, \mathbb{N}_1} \rightarrow \mathrm{Gest}_{\mathrm{Gest}(D(\mathbb{S}))}^{1\text{-et}}.$$

As an important example, we can identify the internal real number object.

PROPOSITION 9.1. *The internal real number object of Cauchy reals in $\mathrm{Gest}_{\mathrm{Gest}(D(\mathbb{S}))}$ is given by the image of the real numbers $\mathbb{R} \in \mathrm{CondAni}^{\mathrm{light}}$ and is 1-affine, equal to*

$$\mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, D(\mathbb{S}))).$$

PROOF. It is clear that the internal construction of light profinite sets yields the functor $\mathrm{ProFin}^{\mathrm{light}} \rightarrow \mathrm{Gest}$ used above. Moreover, $\mathbb{R} \in \mathrm{CondAni}^{\mathrm{light}}$ gets sent to $\mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, D(\mathbb{S})))$, using descendability of $f_*\mathbb{S}$ for a locally finite cover $f : C \rightarrow \mathbb{R}$ of $\mathbb{R}$ by a disjoint union of Cantor sets.²⁴

Now let Cauchy be the internal object of Cauchy sequences of rational numbers, with fixed convergence; i.e. sequences $(x_1, x_2, \dots)$ of rational numbers with $|x_i - x_j| \leq 1/\min(i, j)$ for all i and j . This is closed inside $\prod_{\mathbb{N}} \mathbb{Q}$, and the latter is 1-affine as a countable limit of 1-affine Gestalten. In fact, it is precisely the Gestalt corresponding to the category of $D(\mathbb{S})$ -valued sheaves on the topological space of Cauchy sequences. Using this, one can construct a map

$$\mathrm{Cauchy} \rightarrow \mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, D(\mathbb{S}))).$$

This is surjective, as the map $f : C \rightarrow \mathbb{R}$ used above factors over Cauchy. Moreover, the preimage of $0 \in \mathbb{R}$ is precisely the internal object $\mathrm{Null} \subset \mathrm{Cauchy}$ of nullsequences (automatically with fixed convergence, $|x_i| \leq 1/i$). Thus, the map

$$\mathrm{Cauchy}/\mathrm{Null} \rightarrow \mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, D(\mathbb{S})))$$

is an isomorphism, as desired. $\square$

REMARK 9.2. Of course, the internal real numbers object can also be defined over the absolute base. The proof shows that it is given by the image in Gestalten of

$$\mathrm{Gest}(\mathrm{Shv}(\mathrm{Cauchy}, \mathrm{Ani})) \rightarrow \mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, \mathrm{Ani})),$$

²⁴This can be checked most cleanly using a descendability criterion for sheaves due to Clausen, [Cla25, Theorem 6.21].

where here Cauchy denotes the topological space of Cauchy sequences with fixed convergence. We do not know whether this map is surjective before passing to $D(\mathbb{S})$. In the absence of surjectivity, the image is a priori only given by a 2-étale injection into $\text{Gest}(\text{Shv}(\mathbb{R}, \text{Ani}))$, hence might only be 3-affine.

One can build more general examples by taking more general presentable symmetric monoidal $(\infty, 1)$ -categories. Let us mention two specific examples, already mentioned in the introductory lecture.

- (i) Almost rings yield examples, thus incorporating “almost mathematics”. A related example is Tamarkin’s category $\mathbb{W}$ of completely and continuously $\mathbb{R}$ -filtered $\mathbb{S}$ -modules, cf. [Sch25c]. As shown there, the map

$$\text{Gest}(\mathbb{W}) \rightarrow \text{Gest}(D(\mathbb{S}))$$

is an $\mathbb{R}_{>0}$ -torsor (where $\mathbb{R}_{>0} = \text{Gest}(\text{Shv}(\mathbb{R}_{>0}, D(\mathbb{S})))$ denotes the internal positive real number object), where $\mathbb{R}_{>0}$ acts on $\mathbb{W}$ by multiplicatively rescaling the filtration. In other words, working over $\text{Gest}(D(\mathbb{S}))$, there is a nontrivial map

$$* \rightarrow */\mathbb{R}_{>0}$$

showing that the base point has highly nontrivial internal structure.

- (ii) Analytic rings yield examples. In fact, any analytic ring $A = (A^\flat, D(A))$ consisting of a light condensed ring $A^\flat$ and a certain subcategory $D(A) \subset D(A^\flat)$ of “complete” modules defines a symmetric monoidal localization $D(A^\flat) \rightarrow D(A)$. This yields a functor

$$\text{AnRing} \rightarrow \text{StRing} : A \mapsto (D(A), 1\text{Pr}_{D(A)}, 2\text{Pr}_{D(A)}, \dots).$$

Moreover, analytic stacks are defined precisely as to ensure that this extends to a functor

$$\text{AnStack}^{\aleph_1} \rightarrow \text{Gest}.$$

In general, these are only colimits of 1-affine Gestalten, thus only 2-étale. In practice, fixing some base analytic ring A , the stacks considered are however 1-étale over $\text{Gest}(D(A))$.

How can one give interesting examples of Gestalten that go beyond colimits of 1-affines? It turns out that a natural source are six-functor formalisms.

Namely, say C is a small $(\infty, 1)$ -category with finite limits, and

$$D : \text{Corr} \rightarrow 1\text{Pr}$$

is a lax symmetric monoidal functor, i.e. an $\aleph_1$ -presentable six-functor formalism. Associated to D , there is an interesting symmetric monoidal $(\infty, 2)$ -category K_D of kernels. This is usually described as follows: The objects are the objects of C , and for $X, Y \in C$, one has

$$\text{Hom}_{K_D}(X, Y) = D(X \times Y).$$

Composition is then given by convolution: For $M \in D(X \times Y)$ and $N \in D(Y \times Z)$, their composition is given by

$$\pi_{13}!(\pi_{12}^*M \otimes \pi_{23}^*N)$$

where π_{ij} are the various projections from $X \times Y \times Z$. The identity $1 \in \text{Hom}_{K_D}(X, X)$ is given by $\Delta_X!1$ for the diagonal $\Delta_X : X \rightarrow X \times X$.

A precise construction is usually given by using the theory of enriched $(\infty, 1)$ -categories, and in particular the notion of transfer of enrichment. Namely, $\text{Corr}(C)$ is a closed symmetric monoidal

$(\infty, 1)$ -category (with internal Hom from X to Y also given by $X \times Y$) and hence enriched in itself. One can then transfer enrichment along the lax symmetric monoidal functor D to get an enrichment of $\text{Corr}(C)$ in 1Pr , or just (large) Cat_∞ . Defining $(\infty, 2)$ -categories as being $(\infty, 1)$ -categories enriched in $(\infty, 1)$ -categories, this yields the desired $(\infty, 2)$ -category.

However, there is a second construction that works directly within the world of presentable $(\infty, 2)$ -categories that we use here, and I find it even simpler. Namely, consider the free symmetric monoidal presentable $(\infty, 2)$ -category on $\text{Corr}(C)^{\text{op}}$. By Yoneda, this is

$$\text{Fun}(\text{Corr}(C), \text{Ani}) \otimes_{\text{Ani}} 1\text{Pr} = \text{Fun}(\text{Corr}(C), 1\text{Pr}).$$

By its definition, this is naturally symmetric monoidal; the symmetric monoidal structure is given by Day convolution. Now recall that in the Day convolution symmetric monoidal structure, the commutative algebra objects are precisely the lax symmetric monoidal functors. Thus,

$$D \in \text{CAlg}(\text{Fun}(\text{Corr}(C), 1\text{Pr}))$$

and we can pass to modules over D ,

$$K_D := \text{Mod}_D \text{Fun}(\text{Corr}(C), 1\text{Pr}).$$

This comes with a symmetric monoidal functor from $\text{Corr}(C)^{\text{op}}$, as the composite

$$\text{Corr}(C)^{\text{op}} \rightarrow \text{Fun}(\text{Corr}(C), 1\text{Pr}) \rightarrow \text{Mod}_D \text{Fun}(\text{Corr}(C), 1\text{Pr}) = K_D.$$

We denote this by $X \mapsto [X]_{K_D}$. This is a free construction so it is easy to map out of; explicitly, for any $X \in \text{Corr}(C)^{\text{op}}$ and $M \in K_D$, one has $\text{Hom}_{K_D}([X]_{K_D}, M) = M(X)$. In particular, as the unit object of K_D is given by D itself, we have

$$\text{Hom}_{K_D}([X]_{K_D}, 1) = D(X).$$

As all objects of $\text{Corr}(C)$ are canonically selfdual, one gets that in general

$$\text{Hom}_{K_D}([X]_{K_D}, [Y]_{K_D}) = D(X \times Y)$$

and one recovers the naive recipe above. Identifying enriched $(\infty, 1)$ -categories with marked presentable module categories, the two constructions agree; this is due to Dauser, and the argument is recorded in **[Aok25b]**.

Thus, starting from D we have constructed

$$K_D \in \text{CAlg}(2\text{Pr}),$$

giving an interesting example of a presentable symmetric monoidal $(\infty, 2)$ -category. In fact, if one would set oneself the task of constructing such an object, one would end up to something close to the above: Namely, one would start with some generating small $(\infty, 1)$ -category C of objects, endowed with some symmetric monoidal structure. Afterwards, one can pass to modules over some commutative algebra in there. Proceeding in this way, one can define all the examples where the unit is 1Pr -atomic and which are generated by 1-dualizable objects — one can then take C to be the 1-dualizable objects.

When the categorical Künneth formula holds, the resulting K_D turns out to be “boring”:

PROPOSITION 9.3. *The natural map*

$$\text{Mod}_{D(*)}(1\text{Pr}) \rightarrow K_D$$

(adjoint to $D(*) = \text{End}_{K_D}(1)$) is fully faithful. An object $M \in K_D$ lies in the essential image if and only if for all $X \in C$ the natural map

$$D(X) \otimes_{D(*)} M(*) \rightarrow M(X)$$

is an isomorphism.

In particular, $K_D = \text{Mod}_{D(*)}(\mathbf{1Pr})$ if and only if for all $X, Y \in C$ the functor

$$D(X) \otimes_{D(*)} D(Y) \rightarrow D(X \times Y)$$

is an isomorphism.

Objects in the essential image of

$$\text{Mod}_{D(*)}(\mathbf{1Pr}) \rightarrow K_D$$

are also known as “restricted” in the geometric Langlands literature, cf. e.g. [Gai25, 2.1.5].

PROOF. From its construction, the unit $1 \in K_D$ is 1Pr-atomic (as this property is stable under passing to module categories, and true for generating objects of free 1Pr-linear categories); this implies that the right adjoint $K_D \rightarrow \text{Mod}_{D(*)}(\mathbf{1Pr})$ is 1Pr-linear, and hence even $\text{Mod}_{D(*)}(\mathbf{1Pr})$ -linear. As it is fully faithful on the unit, it follows that $\text{Mod}_{D(*)}(\mathbf{1Pr}) \rightarrow K_D$ is fully faithful. Moreover, evaluating the counit of the adjunction on $M \in K_D$, one gets the transformation $D(X) \otimes_{D(*)} M(*) \rightarrow M(X)$ of functors $\text{Corr}(C) \rightarrow \mathbf{1Pr}$, so $M \in K_D$ lies in the image if and only if this map is an isomorphism for all $X \in C$. $\square$

For any $X \in C$, we can repeat the above construction with the slice $C_{/X}$, defining

$$K_{D,X} = \text{Mod}_D \text{Fun}(\text{Corr}(C_{/X}), \mathbf{1Pr}) \in \text{CAlg}(2\text{Pr}).$$

We note here that the pullback functor $\text{Corr}(C) \rightarrow \text{Corr}(C_{/X})$ induces, by passing to free presentable $(\infty, 2)$ -categories, a symmetric monoidal functor

$$\text{Fun}(\text{Corr}(C), \mathbf{1Pr}) \rightarrow \text{Fun}(\text{Corr}(C_{/X}), \mathbf{1Pr})$$

and this takes D to $D|_{C_{/X}}$; in fact, this functor takes any F to $F|_{C_{/X}}$, as $\text{Corr}(C) \rightarrow \text{Corr}(C_{/X})$ has a left adjoint taking $Y \in C_{/X}$ to the underlying object $Y \in C$. By this base-change compatibility of D , there is only one sense of “passing to modules over D ”.

For any map $f : Y \rightarrow X$ in C , the pullback functor $f^* : C_{/X} \rightarrow C_{/Y}$ induces a symmetric monoidal functor

$$f_1^* : K_{D,X} \rightarrow K_{D,Y},$$

so this assembles into a functor

$$C^{\text{op}} \rightarrow \text{CAlg}(2\text{Pr}).$$

PROPOSITION 9.4. *For any map $f : Y \rightarrow X$, the functor*

$$f_1^* : K_{D,X} \rightarrow K_{D,Y}$$

admits isomorphic left and right adjoints $f_{1\sharp} \cong f_{1}$, whose formation commutes with base change.*

PROOF. The pullback functor $\text{Corr}(C_{/X}) \rightarrow \text{Corr}(C_{/Y})$ admits a simultaneous left and right adjoint induced by the forgetful functor $C_{/Y} \rightarrow C_{/X}$. This passes through the whole construction, using that “passing to modules over D ” is compatible with base change as noted above. $\square$

Thus, we get a functor $C \rightarrow \text{Gest}$ taking $X \in C$ to the 2-affine Gestalt $\text{Gest}(K_{D,X})$. Being 2-affine, all maps are 2-proper, but in fact also 1-prim by the previous proposition. We would also like to say that they are 1-suave, but the existence of left adjoints for the pullback map $\text{Mod}_{K_{D,X}}(2\text{Pr}) \rightarrow \text{Mod}_{K_{D,Y}}(2\text{Pr})$ is not clear. In fact, we would like to say that everything is 1-proper and 1-étale, by applying the proposition also to diagonals. However, the Künneth formula may fail.

For this reason, we repeat the constructions, to define also presentable symmetric monoidal (∞, n) -categories for all $n \geq 2$. Namely, note that

$$C^{\text{op}} \rightarrow \text{CAlg}(2\text{Pr}) : X \mapsto K_{D,X}$$

admits left adjoints for all maps, satisfying base change, and hence upgrades to a six-functor formalism

$$\text{Corr}(C) \rightarrow 2\text{Pr} = \text{Mod}_{1\text{Pr}}(1\text{Pr}) \rightarrow 1\text{Pr}$$

We can then repeat the constructions above with this once-categorified sheaf theory, and iterate; this defines a Stefanich ring²⁵

$$A_D = (D(*), K_D, \dots).$$

In fact, we can also repeat the construction for any slice C/X , yielding a Stefanich ring

$$A_{D,X} = (D(X), K_{D,X}, \dots)$$

functorial in X , so a functor

$$C^{\text{op}} \rightarrow \text{StRing}.$$

Everything is now well-behaved:

PROPOSITION 9.5. *The functor*

$$C \rightarrow \text{Gest}/\text{Gest}(A_D) : X \mapsto [X]_D := \text{Gest}(A_{D,X})$$

commutes with finite limits. For any map $f : Y \rightarrow X$ in C , the corresponding map $[f]_D : [Y]_D \rightarrow [X]_D$ is 1-étale and 1-proper, and comes with an isomorphism $f_{1\sharp} \cong f_{1}$ between left and right adjoints to pullback*

$$f_1^* : K_{D,X} \rightarrow K_{D,Y}.$$

The composite

$$C^{\text{op}} \xrightarrow{[-]_D} \text{Gest}^{\text{op}} \xrightarrow{\mathcal{O}(-)_1} \text{CAlg}(1\text{Pr})$$

recovers the original functor D .

In the theory of six functors, there is a general heuristic idea that often, six-functor formalisms can be written as a composite of a “transmutation” functor from C into some category of geometric objects like stacks, together with the functor of taking quasicohherent sheaves. This proposition says that this is always possible, and in fact completely canonical, when taking Gestalten as the notion of geometric objects.

²⁵This construction is originally due to Stefanich, [Ste25a].

PROOF. First, for any map $f : Y \rightarrow X$ in C , the map $[f]_D : [Y]_D \rightarrow [X]_D$ is 1-prim and 1-suave, as at each categorical level Proposition 9.4 applies. (The extra condition in being n -suave is automatic by Lemma 6.22.) This argument also gives the identification of left and right adjoints. For the rest, it is sufficient to prove commutation with finite limits as this will then prove 1-properness and 1-étaleness by passing to diagonals. It suffices to prove commutation with fibre products; passing to slices, it suffices to prove commutation with binary products. But this follows from the Künneth formula for lower- $*$ — here we use that the formation of these (left and) right adjoints commutes with base change in C , which ensures that for each n ,

$$C^{\text{op}} \rightarrow \text{CAlg}(n\text{Pr}) : X \mapsto \mathcal{O}(A_{D,X})_n$$

defines a six-functor formalism with respect to the right adjoints. $\square$

REMARK 9.6. Under this transmutation $C \rightarrow \text{Gest} : X \mapsto [X]_D$, notions from six-functor formalisms correspond to notions for Gestalten. More precisely, let $f : Y \rightarrow X$ be a map in C . Then:

- (1) The map f is suave if and only if $[f]_D$ is 0-suave.
- (2) The map f is prim if and only if $[f]_D$ is 0-prim.
- (3) The map f is cohomologically étale if and only if f is truncated and $[f]_D$ is 0-étale.
- (4) The map f is cohomologically proper if and only if f is truncated and $[f]_D$ is 0-proper.

(We see that it might have been better to define “cohomologically étale” and “cohomologically proper” in a different way, not asking for truncatedness.) All statements follow directly from the definitions.

The extension to stacks is now also automatic. In fact, recall that in [Sch25b, Definition 5.3], a Grothendieck topology on C is introduced, called the D -topology.

PROPOSITION 9.7. *Assume that $\{f_i : Y_i \rightarrow X\}$ satisfy universal $!$ -descent, i.e. after any pullback $X' \rightarrow X$ in C , the upper- $!$ -functor*

$$D(X') \rightarrow \lim_{\Delta}^! \left(\prod_i D(Y_i \times_X X') \rightrightarrows \prod_{i,j} D(Y_i \times_X Y_j \times_X X') \dots \right)$$

is an equivalence. Then $\bigsqcup_i [f_i]_D : \bigsqcup_i [Y_i]_D \rightarrow [X]_D$ is a cover, i.e. effective epimorphism, in Gestalten.

PROOF. We can assume that X is the final object of C . Assume first that I is countable; we will argue later that we can reduce to this case. Then as all $[f_i]_D$, and hence their disjoint union, is 1-étale, we need to check that the natural map

$$\text{colim}_! \left(\dots \bigsqcup_{i,j} [Y_i \times_X Y_j]_{K_D} \rightrightarrows \bigsqcup_i [Y_i]_{K_D} \right) \rightarrow [*]_{K_D} = 1$$

in $K_D = \mathcal{O}([*]_D)_2$ is an isomorphism, cf. Proposition 8.10. But as K_D is 1Pr-atomically generated by the $[X']_{K_D}$ for $X' \in C$, this is equivalent to checking this on maps from those X' ; and this yields the $!$ -descent condition.

In general, even if I is not countable, the same argument shows that the map

$$\operatorname{colim}! \left(\dots \bigsqcup_{i,j} [Y_i \times_X Y_j]_{K_D} \rightrightarrows \bigsqcup_i [Y_i]_{K_D} \right) \rightarrow [*]_{K_D} = 1$$

in K_D is an isomorphism. But $1 \in K_D$ is countably compact, so it follows that this already happens for some countable subset $I' \subset I$, reducing us to the countable case. $\square$

In particular, we get the following corollary.

COROLLARY 9.8. *The functor $C \rightarrow \operatorname{Gest}_{/[*]_D} : X \mapsto [X]_D$ extends to a functor*

$$\operatorname{Shv}_D(C)^{\aleph_1} \rightarrow \operatorname{Gest}_{/[*]_D}^{1\text{-et}} : X \mapsto [X]_D$$

that commutes with countable colimits and finite limits.

Thus, in thinking and defining the D -topology corresponding to a six-functor formalism, one was actually secretly already seeing the canonical topology on $\operatorname{Gestalten}$.

A natural question arises here, whether one can in fact recover the full six-functor formalism, including the $!$ -functors, from the transmutation into $\operatorname{Gestalten}$ — what is certainly true is that the pullback and tensor functorialities arise as

$$\operatorname{Shv}_D(C)^{\aleph_1, \operatorname{op}} \xrightarrow{[-]_D} \operatorname{Gest}^{\operatorname{op}} \xrightarrow{\mathcal{O}(-)_1} \operatorname{CAlg}(1\operatorname{Pr}).$$

We will take up this question in the appendix, discussing how one can define a class of $!$ -able maps of $\operatorname{Gestalten}$, and a six-functor formalism there. The results will not be definitive, however.

Let us end this lecture by explaining some results of Aoki [Aok25b]. For this, we specialize to the case that C is the $(1, 1)$ -category of separated schemes of finite type over $\mathbb{Z}$. In this context, one usually studies six-functor formalisms like étale cohomology or Betti cohomology taking values in stable $(\infty, 1)$ -categories, and satisfy excision, $\mathbb{A}^1$ -invariance, etc. . By a theorem of Drew–Gallauer [DG22], there is in fact an initial such six-functor formalism, namely Morel–Voevodsky’s SH . Applying the constructions above, we get a $\operatorname{Gestalt}$

$$[\operatorname{Spec}(\mathbb{Z})]_{\operatorname{SH}} \in \operatorname{Gest},$$

and a transmutation functor

$$C \rightarrow \operatorname{Gest}_{[\operatorname{Spec}(\mathbb{Z})]_{\operatorname{SH}}} : X \mapsto [X]_{\operatorname{SH}},$$

extending also to stacks. The following result was first observed in [Sch24].

PROPOSITION 9.9. *The $\operatorname{Gestalt}$ $[\operatorname{Spec}(\mathbb{Z})]_{\operatorname{SH}}$ is 2-affine, hence*

$$[\operatorname{Spec}(\mathbb{Z})]_{\operatorname{SH}} = \operatorname{Gest}(K_{\operatorname{SH}}).$$

Moreover, K_{SH} is generated, as a presentable $(\infty, 2)$ -category, by the objects $[\mathbb{A}_{\mathbb{Z}}^n]_{K_{\operatorname{SH}}}$, $n \geq 0$.

For any $X \in C$, the map

$$[X]_{\operatorname{SH}} \rightarrow [\operatorname{Spec}(\mathbb{Z})]_{\operatorname{SH}}$$

is 1-affine.

PROOF. It suffices to see that for all $X \in C$, denoting $f : [X]_{\text{SH}} \rightarrow [\text{Spec}(\mathbb{Z})]_{\text{SH}}$ the projection, the natural map

$$K_{\text{SH},X} \rightarrow \text{Mod}_{f_{1*}1} K_{\text{SH}}$$

is an equivalence. Indeed, this implies that Proposition 9.3 applies to the six-functor formalism $X \mapsto K_{\text{SH},X}$ to show that its next category of kernels is simply given by passing to modules, and then this property will inductively pass to all higher levels.

To prove this, it is enough to prove that the image of $K_{\text{SH}} \rightarrow K_{\text{SH},X}$ generates the target as a presentable $(\infty, 2)$ -category. For this, we prove that for any X , the objects $[\mathbb{A}_X^n]_{K_{\text{SH},X}}$ generate $K_{\text{SH},X}$. By its definition, the objects $[Y]_{K_{\text{SH},X}}$ for $Y \in C_{/X}$ generate. Covering Y by affines, we can assume that Y is affine. But then Y admits a closed immersion into $\mathbb{A}_X^n$ for some n (using that X is separated), and for a closed immersion $i : Y \hookrightarrow \mathbb{A}_X^n$, one has that $[Y]_{K_{\text{SH},X}}$ is a retract of $[\mathbb{A}_X^n]_{K_{\text{SH},X}}$ via i_* and i^* . $\square$

In particular, while $\text{SH}(\text{Spec}(\mathbb{Z}))$ requires all smooth $\mathbb{Z}$ -schemes as generators (or at least a large class of them), one categorical level higher, just the affine spaces $\mathbb{A}^n$ suffice.

Recall that six-functor formalisms are also often encoded in ring stacks. In the present language, it is better to say “Ringgestalten”, and in fact

$$[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$$

is a Ringgestalt over $[\text{Spec}(\mathbb{Z})]_{\text{SH}}$ — indeed, the ring structure of $\mathbb{A}_{\mathbb{Z}}^1$ transmutes, under the finite-limit preserving functor $X \mapsto [X]_{\text{SH}}$, to a ring structure on $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$. Also note that, again by being finite-limits preserving, this is automatically a 0-truncated object in $\text{Gest}_{[\text{Spec}(\mathbb{Z})]_{\text{SH}}}$, and hence a “static” ring.

From the previous proposition, it is easy to deduce that $[\text{Spec}(\mathbb{Z})]_{\text{SH}}$, equipped with the Ringgestalt $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$, injects into the moduli space of Ringgestalten, thus giving a precise sense in which the whole theory of motives comes from Ringgestalten. Aoki [Aok25b] proves a more precise result. In its formulation, we use the six-functor formalism on Gestalten defined in the appendix to this lecture.

THEOREM 9.10. *The Gestalt $[\text{Spec}(\mathbb{Z})]_{\text{SH}}$ is the moduli-Gestalt over $\text{Gest}(D(\mathbb{S}))$ taking a Gestalt S to the anima of Ringgestalten*

$$R \in \text{CRing}(\text{Gest}_{/S})$$

satisfying:

- (1) *The Gestalt R over S is 1-affine, 0-suave with invertible dualizing complex (i.e., “cohomologically smooth”), and homologically contractible.*
- (2) *The map $i : \{0\} \rightarrow R$ is a closed immersion, the map $j : R^\times \rightarrow R$ is an open immersion, and together they form a stratification of R , i.e. are each other’s complements in R .*

REMARK 9.11. The condition that i is a closed immersion implies in particular that it is injective; this implies that the diagonal of R is an injection, and hence R is 0-truncated. It also implies that the diagonal of R is 0-affine and in particular 1-étale; together with the condition of being 0-suave, this implies that R is 1-étale. As it is also 1-affine, it is 1-proper and 1-étale, as well as truncated; thus it is !-able in the sense of the appendix. Thus, the dualizing complex is well-defined, which is then used in (1).

The condition of being homologically contractible is just the condition that $f_{0\sharp}1 \rightarrow 1$ is an isomorphism, where $f : R \rightarrow S$ is the projection; equivalently, f_0^* is fully faithful.

Recall that a longstanding dream of mine, cf. [Sch18], is that there is a “theory of shtukas over $\mathrm{Spec}(\mathbb{Z})$ ”. This should roughly take the following form. For any scheme S (or, probably, some more general class of spaces), there is an associated geometric object “ $S \times \mathrm{Spec}(\mathbb{Z})$ ”. This should extend the constructions of [SW20] which to a perfectoid space S of characteristic p associate an object “ $S \times \mathrm{Spa}(\mathbb{Z}_p)$ ”, an incarnation of the Fargues–Fontaine curve. In the p -adic case, this comes with a Frobenius lift φ_p ; following Deninger, a reasonable analogue for $S \times \mathrm{Spec}(\mathbb{Z})$ would be the existence of a $\mathbb{R}_{\geq 1}$ -action (which when evaluated on $p \in \mathbb{R}_{\geq 1}$ is the Frobenius lift φ_p); here, we are implicitly modifying the p -adic part of the picture from “ $S \times \mathrm{Spa}(\mathbb{Z}_p)$ ” to

$$(\text{“}S \times \mathrm{Spa}(\mathbb{Z}_p)\text{”} \times \mathbb{R}_{>0})/(\varphi_p \times p^{-1})^{\mathbb{Z}}.$$

(Similar modifications arise quite naturally in the study of analytic syntomifications.)

A $\mathrm{Spec}(\mathbb{Z})$ -shtuka should then be a vector bundle on “ $S \times \mathrm{Spec}(\mathbb{Z})$ ” suitably equivariant with respect to the $\mathbb{R}_{\geq 1}$ -action, possibly with some zeroes/poles. One basic expectation, extrapolated from the close relation between prismatic cohomology and p -adic shtukas, is that

“The universal cohomology theory for algebraic varieties takes values in $\mathrm{Spec}(\mathbb{Z})$ -shtukas”

which should concretely mean that there is a functor from motives over S to quasicoherent sheaves on “ $S \times \mathrm{Spec}(\mathbb{Z})$ ”.

Now note that the functor $S \mapsto [S]_{\mathrm{SH}}$ satisfies some of the basic desiderata for the association $S \mapsto \text{“}S \times \mathrm{Spec}(\mathbb{Z})\text{”}$! Namely, Gestalten form a class of geometric objects (as they form a topos) and come with a notion of quasicoherent sheaves (by taking X to $\mathcal{O}(X)_1 \in \mathrm{CAlg}(1\mathrm{Pr})$), so that in fact motives, in the form of $\mathrm{SH}(S)$, are precisely quasicoherent sheaves on $[S]_{\mathrm{SH}}$.

However, I would also want “ $S \times \mathrm{Spec}(\mathbb{Z})$ ” to be a “nice” geometric object. In the present formalism, this translates into the question about the nature of the Gestalt $[S]_{\mathrm{SH}}$. The map $[S]_{\mathrm{SH}} \rightarrow [\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}}$ is in fact nice as its properties come via transmutation from those of S over $\mathrm{Spec}(\mathbb{Z})$. However, the base $[\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}}$ is not nice, geometrically; one can check that while it is 1-proper, it is not 1-étale over $\mathrm{Gest}(D(\mathbb{S}))$. (Otherwise, for $f : [\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}} \rightarrow \mathrm{Gest}(D(\mathbb{S}))$, one has an isomorphism $f_{1\sharp} \cong f_{1*}$, but $f_{1\sharp}$ takes any $[X]_{K_{\mathrm{SH}}}$ to the dual of the compactly generated $\mathrm{SH}(X)$, and this is not functorially equivalent to $\mathrm{SH}(X)$.)

The key issue is that while for any X , $\mathrm{SH}(X)$ is compactly generated and hence dualizable, and also selfdual, using Poincaré duality on the compact objects (which are the constructible complexes), the selfduality is not the one coming from the natural pairing

$$\mathrm{SH}(X) \otimes \mathrm{SH}(X) \xrightarrow{\otimes} \mathrm{SH}(X) \xrightarrow{-!} \mathrm{SH}(\mathrm{Spec}(\mathbb{Z})).$$

In the formalism of Betti sheaves, things are better. More generally, working with Berkovich motives [Sch25a], things are better — here, we take C to be the category of Berkovich spaces “of finite type” over $\mathbb{Z}$ (say, closed in a polyannulus over $\mathbb{Z}$). For any $X \in C$, a theorem of [Sch25a] shows that the category of Berkovich motives $D_{\mathrm{mot}}(X) \in \mathrm{CAlg}(1\mathrm{Pr})$ is 1-rigid, and in particular, selfdual with respect to the natural pairing.²⁶ In fact, Aoki proves:²⁷

THEOREM 9.12. *Let $[\mathcal{M}(\mathbb{Z})]_{\mathrm{Berk}}$ over $\mathrm{Gest}(D(\mathbb{Z}))$ denote the Gestalt corresponding to the six-functor formalism from [Sch25a] restricted to the finite-type objects C above. Then*

²⁶Like Betti sheaves, it is however not compactly generated! Namely, it is closely related to sheaves on the compact Hausdorff space underlying the Berkovich space.

²⁷He even considers the situation with $D(\mathbb{S})$ -coefficients.

- (1) *It is the moduli-Gestalt parametrizing, over a Gestalt S , Ringgestalten*

$$R \in \mathbf{CRing}(\mathbf{Gest}_{/S})$$

satisfying the conditions from Theorem 9.10, as well as the condition that the Kummer and Artin–Schreier maps

$$R[\frac{1}{p}]^\times \xrightarrow{x \mapsto x^p} R[\frac{1}{p}]^\times, R/p \xrightarrow{x \mapsto x - x^p} R/p$$

are surjective for all primes p ; and equipped with a norm map

$$N : R \rightarrow \mathbb{R}_{\geq 0}$$

(where $\mathbb{R}_{\geq 0} \in \mathbf{Gest}_{/S}$ denotes the internal nonnegative real numbers object) which is multiplicative and satisfies the triangle inequality, with the disk $N^{-1}([0, 1])$ having trivial cohomology, and $N^{-1}(0) = \{0\} \subset R$.

- (2) *Up to an issue at the prime 2, the Gestalt $[\mathcal{M}(\mathbb{Z})]_{\text{Berk}} \rightarrow \mathbf{Gest}(D(\mathbb{Z}))$ is 0-proper. More precisely, it is 1-étale, its diagonal is 0-proper, it is 1-prim, and after base change to $\mathbb{Z}[\frac{1}{2}]$ it is 0-proper. Similarly, $[\mathcal{M}(\mathbb{Z}[\sqrt{-1}])]_{\text{Berk}} \rightarrow \mathbf{Gest}(D(\mathbb{Z}))$ is 0-proper.*²⁸

REMARK 9.13. The theory of Berkovich motives has so far only been developed in the étale setting. Its non-étale version should amount to omitting the Kummer and Artin–Schreier conditions above. In fact, Aoki shows that in the algebraic setting, the passage from non-étale to étale motives amounts to enforcing those conditions.

The 0-properness in part (2) translates into a “2-rigidity” of the corresponding presentable symmetric monoidal $(\infty, 2)$ -category of kernels. It is probably the first nontrivial example of such an object.

Thus, $[\mathcal{M}(\mathbb{Z})]_{\text{Berk}}$, or more generally $[S]_{\text{Berk}}$ for a Berkovich space S , forms a slightly more “geometric” incarnation of a space like “ $S \times \text{Spec}(\mathbb{Z})$ ”. It has the additional feature of admitting a natural $\mathbb{R}_{\geq 1}$ -action: Indeed, one can simply rescale the norm. On the subspace

$$[\mathcal{M}(\mathbb{F}_p)]_{\text{Berk}} \subset [\mathcal{M}(\mathbb{Z})]_{\text{Berk}}$$

where the universal Ringgestalt R is an $\mathbb{F}_p$ -algebra, the action of $\mathbb{R}_{\geq 1}$ factors over $\mathbb{R}_{\geq 1}/p^{\mathbb{N}}$. Indeed, this R itself carries a Frobenius endomorphism $F : R \rightarrow R$ which in fact is an isomorphism, but rescales the norm by a factor of p . In other words, rescaling the norm by a p -th power is naturally equivalent to the identity functor, via the isomorphism $F : R \rightarrow R$.

In summary, in the world of Gestalten, we can define a natural geometric object $[\mathcal{M}(\mathbb{Z})]_{\text{Berk}}$ with a “Frobenius flow” action of $\mathbb{R}_{\geq 1} \cong^{\log} \mathbb{R}_{\geq 0}$, such that each prime p yields a closed subspace $[\mathcal{M}(\mathbb{F}_p)]_{\text{Berk}}$ on which the flow is periodic, with period $p \in \mathbb{R}_{\geq 1}$ (i.e., $\log(p) \in \mathbb{R}_{\geq 0}$).

This still does not realize the dream of shtukas over $\text{Spec}(\mathbb{Z})$, but it does seem like a step forward.

²⁸The only issue here is the infinite cohomological dimension at the prime 2 of $\text{Gal}(\mathbb{C}/\mathbb{R})$.

Appendix to Lecture IX: Six Functors on Gestalten

Above, we had seen that for any six-functor formalism D on a small $(\infty, 1)$ -category C with finite limits, maps f in C yield maps of Gestalten that are 1-étale and 1-proper, and moreover come with an identification $f_{1\sharp} \cong f_{1*}$. This, in fact, yields !-functors: Namely, we get the counit map

$$f_{1*}f_1^* \cong f_{1\sharp}f_1^* \rightarrow \text{id}$$

which, when evaluated at the unit, yields a map

$$f_{0,!} : (\mathcal{O}(Y)/\mathcal{O}(X))_1 \rightarrow 1$$

in $\mathcal{O}(X)_2$. (Said differently, the identification $f_{1\sharp} \cong f_{1*}$ yields an identification $(\mathcal{O}(Y)/\mathcal{O}(X))_1^! \cong (\mathcal{O}(Y)/\mathcal{O}(X))_1$, along which one can transfer $f_{0,!}$.) Taking $\text{Hom}_{\mathcal{O}(X)_2}(1, -)$, this yields in particular a map

$$f_{0,!} : \mathcal{O}(Y)_1 \rightarrow \mathcal{O}(X)_1.$$

This suggests the following definition.

DEFINITION 9.14. A map $f : Y \rightarrow X$ of Gestalten is quasi-!-able if f is 1-étale and 1-proper. The map f is !-able if it is quasi-!-able and some iterated diagonal of f is 0-proper. The map f is co-!-able if it is quasi-!-able and some iterated diagonal of f is 0-étale.

Recall that if functors admit both left and right adjoints, it is important to also control their interaction in terms of double Beck–Chevalley transformations. This is ensured by the following proposition.

PROPOSITION 9.15. *Consider a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & \searrow h & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

of Gestalten, and assume that f is n -suave while g is n -prim. Then the double Beck–Chevalley map

$$f_{n\sharp}g'_{n*} \rightarrow g_{n*}f'_{n\sharp}$$

of maps from $\mathcal{O}(Y')_{n+1}$ to $\mathcal{O}(X)_{n+1}$ is an isomorphism. In fact, it is even an isomorphism of maps from $h_{n+1,}1$ to 1 in $\mathcal{O}(X)_{n+2}$.*

PROOF. First, we claim that there is a natural transformation

$$f_{n+1,*}f_{n+1}^* \xrightarrow{f_{n\sharp}} \text{id}$$

in $\text{End}(\mathcal{O}(X)_{n+2})$. If f is n -étale, so that $f_{n+1,*}$ is linear, this just obtained from $f_{n+1,*}1 \xrightarrow{f_{n\sharp}} 1$ in $\mathcal{O}(X)_{n+2}$ by letting $\mathcal{O}(X)_{n+2}$ act on itself. In general, one first has a natural transformation

$$\text{id} \xrightarrow{f_{n!}} f_{n+1,\sharp}f_{n+1}^*$$

in $\text{End}(\mathcal{O}(X)_{n+2})$, by the similar argument, and then passes to right adjoints.

Now consider the natural morphism $1 \rightarrow g_{n+1,*}1$ in $\mathcal{O}(X)_{n+2}$. Applying $f_{n+1,*}f_{n+1}^*$ to it, this becomes $f_{n+1,*}1 \rightarrow h_{n+1,*}1$, and the existence of $f_{n\sharp}$ amounts to the commutative square

$$\begin{array}{ccc} f_{n+1,*}1 & \longrightarrow & h_{n+1,*}1 \\ f_{n\sharp} \downarrow & & \downarrow f'_{n\sharp} \\ 1 & \longrightarrow & g_{n+1,*}1 \end{array}$$

expressing the base change compatibility of $f_{n\sharp}$.

On the other hand, we can also apply this reasoning to the right adjoint $g_{n*} : g_{n+1,*}1 \rightarrow 1$ of $1 \rightarrow g_{n+1,*}1$. Applying $f_{n+1,*}f_{n+1}^*$ to it, this becomes $g'_{n*} : h_{n+1,*}1 \rightarrow f_{n+1,*}1$, and then $f_{n\sharp} : f_{n+1,*}f_{n+1}^* \rightarrow \text{id}$ in $\text{End}(\mathcal{O}(X)_{n+2})$ gives the desired commutative diagram. $\square$

In particular, we get the following corollary. (In principle one could state this with 1 replaced by any $n \geq 0$.)

COROLLARY 9.16. *Let $f : Y \rightarrow X$ be a map of Gestalten that is 1-étale and 1-proper. Then $f_{1\sharp}, f_{1*} : \mathcal{O}(Y)_2 \rightarrow \mathcal{O}(X)_2$ agree up to twist. More precisely, there are natural isomorphism*

$$f_{1\sharp}(p_{1*}\Delta_{f,1,\sharp}1 \otimes -) \cong f_{1*}, f_{1*}(p_{1\sharp}\Delta_{f,1,*}1 \otimes -) \cong f_{1\sharp}$$

and

$$p_{1*}\Delta_{f,1,\sharp}1 \otimes p_{1\sharp}\Delta_{f,1,*}1 \cong 1 \in \mathcal{O}(X)_2,$$

so both tensor factors are invertible. Here $Y \xrightarrow{\Delta_f} Y \times_X Y \xrightarrow{p} Y$ are the diagonal and first projection.

PROOF. Indeed, using linearity of all intervening functors, as well as the double Beck–Chevalley isomorphism,

$$\begin{aligned} f_{1\sharp}(p_{1*}\Delta_{f,1,\sharp}1 \otimes M) &\cong f_{1\sharp}p_{1*}\Delta_{f,1,\sharp}\Delta_{f,1}^*p_1^*M \\ &\cong f_{1\sharp}p_{1*}\Delta_{f,1,\sharp}M \\ &\cong f_{1*}q_{1\sharp}\Delta_{f,1,\sharp}M \\ &\cong f_{1*}M, \end{aligned}$$

where $q : Y \times_X Y \rightarrow Y$ is the second projection. The other isomorphism is analogous. Moreover,

$$\begin{aligned} p_{1*}\Delta_{f,1,\sharp}1 \otimes p_{1\sharp}\Delta_{f,1,*}1 &\cong p_{1*}\Delta_{f,1,\sharp}\Delta_{f,1}^*p_1^*p_{1\sharp}\Delta_{f,1,*}1 \\ &\cong p_{1*}\Delta_{f,1,\sharp}p_{1\sharp}\Delta_{f,1,*}1 \\ &\cong p_{1*}\Delta_{f,1,*}1 \cong 1. \end{aligned}$$

$\square$

In particular, for any 1-étale and 1-proper $f : Y \rightarrow X$, we get a canonically associated invertible $\mathbb{D}_f \in \mathcal{O}(Y)_2$. If f is 0-étale or 0-proper, then one gets a trivialization $\mathbb{D}_f \cong 1$ by ambidexterity. If one relates f and $\Delta_f : Y \rightarrow Y \times_X Y$, one finds that $\mathbb{D}_f$ and $\mathbb{D}_{\Delta_f}$ are inverse. Thus, if f is !-able or co-!-able, one gets a trivialization $\mathbb{D}_f \cong 1$ by passing to some iterated diagonal.

Once one has such a trivialization, we get the counit map

$$f_{0!} : f_{1*}1 \cong f_{1\sharp}1 \rightarrow 1$$

yielding a map $f_{0!} : \mathcal{O}(Y)_1 \rightarrow \mathcal{O}(X)_1$ compatible with base change and composition. We expect that these define a six-functor formalism on !-able maps (or co-!-able maps), but we were not able to fully resolve all coherence issues.

Restricting to truncated maps, we can however get the desired six-functor formalism:

PROPOSITION 9.17 ([Aok25b]). *Fix some Gestalt X , and let C be the small $(\infty, 1)$ -category of Gestalten over X that are truncated, 1-étale and 1-proper. Then there is a six-functor formalism*

$$\mathrm{Corr}(C) \rightarrow \mathcal{O}(X)_2 : Y \mapsto (\mathcal{O}(Y)/\mathcal{O}(X))_1$$

refining the above construction. In particular, forgetting along $\mathcal{O}(X)_2 \rightarrow 1\mathrm{Pr}$, this yields a six-functor formalism

$$\mathrm{Corr}(C) \rightarrow 1\mathrm{Pr} : Y \mapsto \mathcal{O}(Y)_1.$$

PROOF. We apply the theorem of Cnossen–Lenz–Linskens [CLL25] to $I = P = E$ consisting of all morphisms of C , in the once-categorified six-functor formalism $Y \mapsto \mathcal{O}(Y)_2$ (where all maps in C are 0-étale and 0-proper), and then pass to endomorphisms of the unit. $\square$

A separate question is whether this six-functor formalism on Gestalten restricts to the one on a given six-functor formalism C . We expect that if all maps in C are truncated, then the previous argument can be upgraded to show this agreement. In the more general setting, one first has to solve the coherence issues in constructing the six-functor formalism for (co-)!-able maps of Gestalten.

10. Lecture X: Cartier Duality

As a final topic of these lectures, we want to discuss a very general “Cartier Duality” result in the topos of Gestalten. This will unify and generalize various “Fourier isomorphisms”. As in the first lecture, let us start by recalling various examples of this phenomenon.

EXAMPLE 10.1 (Fourier). In this example, we use $\mathbb{C}$ -coefficients everywhere. One of the most basic instances of a Fourier isomorphism is the isomorphism of Hilbert spaces

$$\begin{aligned} L^2(S^1) &\cong L^2(\mathbb{Z}) \\ \left(x \mapsto \sum_{n \in \mathbb{Z}} a_n \exp(2\pi i n x) \right) &\longleftrightarrow (a_n)_n \\ f(x) &\longmapsto \left(n \mapsto \int_{S^1} f(x) \exp(-2\pi i n x) dx \right), \end{aligned}$$

where we regard $S^1 = \mathbb{R}/\mathbb{Z}$, with coordinate x . If we instead regard $S^1 \subset \mathbb{C}^\times$ with coordinate t , the formulas become

$$\begin{aligned} L^2(S^1) &\cong L^2(\mathbb{Z}) \\ \left(x \mapsto \sum_{n \in \mathbb{Z}} a_n t^n \right) &\longleftrightarrow (a_n)_n \\ f(t) &\longmapsto \left(n \mapsto \int_{S^1} f(t) t^{-n} \frac{dt}{t} \right). \end{aligned}$$

There are various refined, “decompleted” versions of this isomorphism, for example

$$\begin{aligned} C^\infty(S^1) &\cong \{\text{sequences of rapid decay } (a_n)_n\}, \\ \mathcal{D}(S^1) &\cong \{\text{sequences of polynomial growth } (a_n)_n\}. \end{aligned}$$

Here $\mathcal{D}(S^1)$ denotes the dual nuclear Fréchet space of Schwartz distributions on S^1 (i.e., the dual of $C^\infty(S^1)$). The Hilbert space isomorphism can be obtained by passing to Hilbert space completions.

Note that there is also a similar statement at the categorical level: Namely, continuous finite-dimensional representations of S^1 decompose into weight spaces, giving an equivalence

$$\text{Rep}(S^1) \cong \bigoplus_{n \in \mathbb{Z}} \text{Vect}^{\text{fin.dim.}}(\mathbb{C}),$$

where the right-hand side can also be thought of as “finite” sheaves of $\mathbb{C}$ -vector space on the discrete set $\mathbb{Z}$.

EXAMPLE 10.2. The previous example has a more elementary algebraic analogue. Namely, considering $S^1 \subset \mathbb{C}^\times$ as a subset of the multiplicative group $\mathbb{G}_m$, we can even further decomplete $L^2(S^1)$ to the space of Laurent polynomials $\mathbb{C}[t^{\pm 1}]$. This in fact works over any (usual, for the moment) commutative ring R . Namely, for any R we get the multiplicative group $\mathbb{G}_{m,R} = \text{Spec}(R[t^{\pm 1}])$ and an isomorphism

$$\mathcal{O}(\mathbb{G}_{m,R}) = \{\text{sequences of finite support } (a_n)_n\} = \bigoplus_{n \in \mathbb{Z}} R,$$

given by the obvious map $(a_n)_n \mapsto \sum_{n \in \mathbb{Z}} a_n t^n$. Note that the right-hand side can also be understood as the compactly supported global sections (or homology) of the structure sheaf on the scheme $\mathbb{Z} \times \text{Spec}(R)$ (a countable disjoint union of copies of $\text{Spec}(R)$).

In this setting, we can give a cleaner version of the passage to categories. Namely, there is an equivalence of categories

$$D_{\text{qc}}(*/\mathbb{G}_{m,R}) \cong \bigoplus_{n \in \mathbb{Z}} D_{\text{qc}}(\text{Spec}(R)) \cong D_{\text{qc}}(\mathbb{Z} \times \text{Spec}(R))$$

by using that representations of $\mathbb{G}_{m,R}$ decompose according to weights. In fact, dually, there is also an equivalence

$$D_{\text{qc}}(\mathbb{G}_{m,R}) \cong D_{\text{qc}}(*/\mathbb{Z} \times \text{Spec}(R)),$$

as the right-hand side is a representation of $\mathbb{Z}$, i.e. a complex of R -modules equipped with an automorphism, i.e. a complex of $R[t^{\pm 1}]$ -modules. The functors realizing the equivalences can also be given by categorified forms of the formulas defining the usual Fourier transform. Namely, abbreviating $\mathbb{G}_m = \mathbb{G}_{m,R}$ and $\mathbb{Z} = \mathbb{Z} \times \text{Spec}(R)$, consider the diagram

$$\begin{array}{ccc} & */\mathbb{G}_m \times \mathbb{Z} & \xrightarrow{m} */\mathbb{G}_m \\ p \swarrow & & \searrow q \\ */\mathbb{G}_m & & \mathbb{Z}. \end{array}$$

Here m denotes the pairing induced by $\mathbb{G}_m \times \mathbb{Z} \rightarrow \mathbb{G}_m : (t, n) \mapsto t^n$, and p and q are the obvious projections. Now on $*/\mathbb{G}_m$ we have a tautological line bundle $\mathcal{L} \in D_{\text{qc}}(*/\mathbb{G}_m)$ corresponding to the tautological representation of $\mathbb{G}_m$, and we can use this (or its inverse, for consistency reasons) as a “kernel” for a “Fourier transform”

$$\begin{aligned} D_{\text{qc}}(*/\mathbb{G}_m) &\rightarrow D_{\text{qc}}(\mathbb{Z}) \\ M &\mapsto q_!(p^*M \otimes m^*\mathcal{L}^{-1}). \end{aligned}$$

Here, $q_!$ plays the role of integrating (over $*/\mathbb{G}_m$), while the thing that is integrated is the analogue of $f(t)t^{-n}$. The inverse functor is similarly given by

$$\begin{aligned} D_{\text{qc}}(*/\mathbb{G}_m) &\leftarrow D_{\text{qc}}(\mathbb{Z}) \\ p_!(q^*M \otimes m^*\mathcal{L}) &\leftarrow M. \end{aligned}$$

(Note that q is proper (in the sense of the quasicohherent six-functor formalism) so $q_! = q_*$, while p is étale so $p_! = p_*$.)

Similarly, we can also consider the diagram

$$\begin{array}{ccc} & \mathbb{G}_m \times */\mathbb{Z} & \xrightarrow{m} */\mathbb{G}_m \\ p \swarrow & & \searrow q \\ \mathbb{G}_m & & */\mathbb{Z} \end{array}$$

and the resulting equivalence

$$\begin{aligned} D_{\text{qc}}(\mathbb{G}_m) &\rightarrow D_{\text{qc}}(*/\mathbb{Z}) \\ M &\mapsto q_!(p^*M \otimes m^*\mathcal{L}^{-1}) \end{aligned}$$

with inverse

$$\begin{aligned} D_{\text{qc}}(\mathbb{G}_m) &\leftarrow D_{\text{qc}}(*/\mathbb{Z}) \\ p_!(q^*M \otimes m^*\mathcal{L}) &\leftarrow M. \end{aligned}$$

(Also in this case, q is proper in the sense of the quasicoherent six-functor formalism, so $q_! = q_*$, while p is étale so $p_! = p_*$.)

We note that the structure sheaf on $\mathbb{G}_m$ corresponds to the regular representation of $\mathbb{Z}$ under this equivalence, and passing to endomorphisms, we recover the Fourier isomorphism on the level of vector spaces. Using analytic stacks over the liquid complex numbers, one can give similar equivalences of categories also in the analytic case, for full derived categories of quasicoherent sheaves on different versions (smooth, real-analytic, ...) of S^1 , categorifying in particular the isomorphisms for C^∞ -functions or Schwartz distributions recalled above.

EXAMPLE 10.3 (Cartier). Let R be, as before, a usual commutative ring. Cartier introduced a duality on finite flat commutative group schemes G . These are of the form $G = \text{Spec}(\mathcal{O}(G))$ where $\mathcal{O}(G)$ is a finite projective R -module equipped with a commutative algebra structure, and a compatible cocommutative coalgebra structure which encodes the addition on G . Some examples are given by $\mathbb{Z}/n$,

$$\mu_n = \ker(\mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m) = \text{Spec}(R[t]/(t^n - 1)),$$

or

$$A[n] = \ker(A \xrightarrow{\times n} A)$$

for an abelian variety A (where n is a positive integer). If R is of characteristic 0, then μ_n is étale locally isomorphic to $\mathbb{Z}/n$; more precisely one has to adjoin a primitive n -th root of unity to R to get an isomorphism. More generally, if R is of characteristic 0, any G is étale locally isomorphic to a constant commutative group. However, if R is of positive or mixed characteristic, finite flat group schemes are very interesting objects.

As $\mathcal{O}(G)$ is both a commutative algebra and a cocommutative coalgebra, the dual R -module $\mathcal{O}(G)^*$ is also a commutative algebra and a cocommutative coalgebra (where the algebra structure comes from the original coalgebra structure, and vice versa). Thus,

$$G^* = \text{Spec}(\mathcal{O}(G)^*)$$

is another finite flat group scheme. This algebraic construction also has a geometric meaning, namely

$$G^* = \mathcal{H}\text{om}(G, \mathbb{G}_m)$$

where the right-hand side denotes the internal Hom in, say, fpqc sheaves. This is induced by a natural pairing

$$m : G \times G^* \rightarrow \mathbb{G}_m$$

which corresponds to the map

$$R[t^{\pm 1}] \rightarrow \mathcal{O}(G) \otimes_R \mathcal{O}(G^*) = \mathcal{O}(G) \otimes_R \mathcal{O}(G)^*$$

sending t to the image of 1 under the coevaluation map $R \rightarrow \mathcal{O}(G) \otimes_R \mathcal{O}(G)^*$ (which turns out to yield an invertible element).

In the examples above, $G = \mathbb{Z}/n$ has dual μ_n , while $G = A[n]$ has dual $G^* = A^*[n]$ where A^* denotes the dual abelian variety. Using the diagram

$$\begin{array}{ccc} & */G \times G^* & \xrightarrow{m} */\mathbb{G}_m \\ p \swarrow & & \searrow q \\ */G & & G^*, \end{array}$$

one can again give “Fourier”-equivalences on quasicoherent sheaves, namely

$$\begin{aligned} D_{\text{qc}}(*/G) &\cong D_{\text{qc}}(G^*) \\ p_*(q^*M \otimes m^*L) &\leftarrow M, \end{aligned}$$

with similar inverse equivalence; and by symmetry also the version with G and G^* exchanged.

EXAMPLE 10.4 (Mukai). The preceding example applies in particular to n -torsion in abelian varieties; but in fact a similar story applies to the abelian varieties themselves. Namely, if A/R is an abelian variety, then there is a dual abelian variety A^*/R ; this can be defined as

$$A^* = \mathcal{E}\text{xt}^1(A, \mathbb{G}_m)$$

where here $\mathcal{E}\text{xt}^1$ denotes the internal Ext^1 in, say, fpqc sheaves. Note that as $\mathbb{G}_m$ is affine but $\mathcal{O}(A) = R$, that $\mathcal{H}\text{om}(A, \mathbb{G}_m) = 0$, so we need to look at the next Ext-group to see something interesting.

In particular, there is a tautological pairing

$$m : A \times A^* \rightarrow */\mathbb{G}_m$$

and hence a line bundle

$$\mathcal{P} = m^*\mathcal{L} \in D_{\text{qc}}(A \times A^*);$$

this is known as the Poincaré line bundle. Mukai observed that, using the diagram

$$\begin{array}{ccc} & A \times A^* & \xrightarrow{m} */\mathbb{G}_m \\ p \swarrow & & \searrow q \\ A & & A^*, \end{array}$$

this induces “Fourier”-equivalences

$$\begin{aligned} D_{\text{qc}}(A) &\cong D_{\text{qc}}(A^*) \\ p_*(q^*M \otimes \mathcal{P}) &\leftarrow M. \end{aligned}$$

Note that in this example, we do not pass to classifying spaces of the abelian varieties.

WARNING 10.5 (Breen). The preceding two examples make it tempting to think that one could in general form duals by taking the derived internal $R\mathcal{H}\text{om}(-, \mathbb{G}_m)$. However, the higher Ext-groups are (probably) very sensitive to the choice of Grothendieck topology, very hard to compute in practice, and in the preceding examples the expected vanishing in degrees > 0 resp. > 1 does not hold. This makes it hard to generalize Cartier duality to “chain complexes of finite flat group schemes” (or of abelian varieties).

EXAMPLE 10.6 (Deligne). In the previous examples, we always worked directly with quasicoherent sheaves. However, there are similar Fourier-equivalences also for other sheaf theories. A famous example is given by the ℓ -adic Fourier transform. For this, we work with varieties over a field k of positive characteristic, and fix a nontrivial additive cocharacter $\psi : \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Using the Artin–Schreier exact sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{A}_k^1 \xrightarrow{x \mapsto x^p - x} \mathbb{A}_k^1 \rightarrow 0,$$

the character ψ yields a 1-dimensional local system $\mathcal{L}_\psi$ on $\mathbb{A}_k^1$, known as the Artin–Schreier sheaf (or “exponential local system”, following an analogy also with the next example). Now if V is any finite-dimensional k -vector space, with dual V^* , and (by abuse of notation) we consider both as affine schemes over k (i.e., if $V = k^d$, we consider it as $\mathbb{A}_k^d$), we get the diagram

$$\begin{array}{ccc} & V \times V^* & \xrightarrow{m} \mathbb{A}_k^1 \\ p \swarrow & & \searrow q \\ V & & V^* \end{array}$$

Then there is an equivalence

$$\begin{aligned} D_{\text{et}}(V, \overline{\mathbb{Q}}_\ell) &\cong D_{\text{et}}(V^*, \overline{\mathbb{Q}}_\ell) \\ p_!(q^* M \otimes m^* \mathcal{L}_\psi) &\leftarrow M, \end{aligned}$$

with similar inverse equivalence.

EXAMPLE 10.7. The previous example admits an analogue in the context of D -modules on varieties over a field k of characteristic 0. Recall that by using Simpson’s de Rham stack, this can be defined by

$$X \mapsto \text{Dmod}(X) = D_{\text{qc}}(X_{\text{dR}}),$$

via importing the six functors from the quasi-coherent six-functor formalism; see [Sch25b, Appendix to Lecture VIII] for a discussion of the six functors from this perspective, and its relation to standard presentations. The role of the Artin–Schreier sheaf is played here by the “exponential D -module” $\mathcal{L}$ on $\mathbb{A}_k^1$ whose solution is the exponential function $e = \exp(t)$. More precisely, $\mathcal{L}$ corresponds to the D -module $k[T] \cdot e$ with $\nabla_{\mathcal{L}}(e) = e$. In fact, this induces a map of commutative groups (in stacks)

$$\mathbb{A}_{\text{dR}}^1 \rightarrow */\mathbb{G}_m$$

which can also be defined more abstractly as the composite

$$\mathbb{A}_{\text{dR}}^1 = \mathbb{A}^1 / \hat{\mathbb{G}}_a \rightarrow */\hat{\mathbb{G}}_a \xrightarrow{\exp} */\mathbb{G}_m.$$

Thus, for a finite-dimensional k -vector space V with dual V^* , we get a diagram

$$\begin{array}{ccccc} & V_{\text{dR}} \times V_{\text{dR}}^* & \xrightarrow{m} & \mathbb{A}_{\text{dR}}^1 & \longrightarrow */\mathbb{G}_m \\ p \swarrow & & & & \\ V_{\text{dR}} & & & & V_{\text{dR}}^* \end{array}$$

and this induces a Fourier equivalence for D -modules

$$\begin{aligned} D_{\text{qc}}(V_{\text{dR}}) &\cong D_{\text{qc}}(V_{\text{dR}}^*) \\ q_!(p^* M \otimes m^* \mathcal{L}) &\leftarrow M. \end{aligned}$$

In fact, for $\mathbb{A}_k^1$, this has an elementary description in terms of actual D -modules: Namely, a D -module on $\mathbb{A}_k^1$ is a module over the Weyl algebra $k\langle t, \nabla \rangle / ([\nabla, t] = 1)$, and the Fourier transform amounts to the automorphism of the Weyl algebra taking t to ∇ and ∇ to $-t$.

There are many more examples of similar dualities:

- (1) working over a field k of characteristic 0, $\mathbb{A}^1$ is Cartier dual to the formal additive group $\hat{\mathbb{G}}_a$;
- (2) working over a nonarchimedean field k of characteristic 0, the analytic affine line $\mathbb{A}^{1, \text{an}}$ is Cartier dual to the overconvergent neighborhood $\mathbb{G}_a^\dagger$ of 0 in $\mathbb{A}^1$ (i.e., functions on $\mathbb{G}_a^\dagger$ are power series converging in some small disk);
- (3) working over $\mathbb{Q}_p$, the open unit disc $\mathbb{D}_{\mathbb{Q}_p}$ is Cartier dual to the locally analytic p -adic numbers $\mathbb{Z}_p^{\text{la}}$ (i.e., functions on $\mathbb{Z}_p^{\text{la}}$ are continuous functions on $\mathbb{Z}_p$ that are locally representable by power series), this is a version of “Amice duality”;
- (4) working directly with symmetric monoidal $(\infty, 1)$ -categories, Lurie has a generalization of Cartier duality in [Lur18a, Section 3]; ...

All of these examples — except the very first one using L^2 -functions — turn out to be special cases of the following general duality. For this, fix any Gestalt X as the base. We can then consider commutative groups (i.e., $\mathbb{E}_\infty$ -groups) in Gestalten over X . As we work in a fully derived situation, this is giving something analogous to the connective part $D_{\geq 0}(\mathbb{S}) \subset D(\mathbb{S})$, and it is better to work stably, and consider spectrum objects in Gestalten over X , i.e.

$$\text{Sp}(\text{Gest}/_X) = \{(M_0, M_1, \dots) \mid M_n \in \text{CGrp}(\text{Gest}/_X), M_n = \Omega M_{n+1}\}^{29}$$

For any $n \geq 0$, we also get the subcategory

$$\text{Sp}(\text{Gest}_{/X}^{n\text{-et}}) \subset \text{Sp}(\text{Gest}/_X)$$

where all M_i are n -étale; this is precisely the $\aleph_1$ -compact objects in sheaves of $\mathbb{S}$ -modules on the actual $(\infty, 1)$ -topos $\text{Ind}_{\aleph_1}(\text{Gest}_{/X}^{n\text{-et}})$. By ambidexterity, we have

$$\bigcup_n \text{Sp}(\text{Gest}_{/X}^{n\text{-et}}) = \bigcup_n \text{Sp}(\text{Gest}_{/X}^{n\text{-prop}}).$$

Moreover, we note that there is a canonical object

$$\text{GL}_1 \in \text{Sp}(\text{Gest}/_X)$$

which represents the functor

$$Y \mapsto (\mathcal{O}(Y)_0^\times, \mathcal{O}(Y)_1^\times, \mathcal{O}(Y)_2^\times, \dots)$$

where $\mathcal{O}(Y)_n^\times \subset \mathcal{O}(Y)_n$ is the subanima of invertible objects, with the $\mathbb{E}_\infty$ -group structure of tensor products. We will actually see that $\text{GL}_1 \in \text{Sp}(\text{Gest}_{/X}^{0\text{-prop}})$.

²⁹We have so far avoided using the term “spectrum”, but do not know a good term to use here otherwise. We could, as before, say $\mathbb{S}$ -module, but at this point we actually want to use this displayed perspective as an infinite sequence of $\mathbb{E}_\infty$ -groups, under delooping. However, note that there is a real language issue: In standard language, we are considering sheaves of spectra on spectra of categorical spectra, where the three occurrences of “spectra” all have a different meaning...

THEOREM 10.8 (S.-Stefanich). *For any $M \in \bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$, the functor*

$$N \mapsto \mathrm{Hom}(N \otimes M, \mathrm{GL}_1)$$

on $\mathrm{Sp}(\mathrm{Gest}_X)$ is representable by an object

$$\mathcal{H}\mathrm{om}(M, \mathrm{GL}_1) \in \bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}}) \subset \mathrm{Sp}(\mathrm{Gest}_X),$$

and this induces an anti-self-equivalence of $\bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$. More precisely, for any $n \geq 0$, the functor $\mathcal{H}\mathrm{om}(-, \mathrm{GL}_1)$ induces an equivalence

$$\mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}}) \cong \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-prop}})^{\mathrm{op}}.$$

Moreover, for any $M \in \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$, any $m \geq n$ and any $k = 0, \dots, m$, we get an equivalence

$$(\mathcal{O}(M_k)/\mathcal{O}(X))_m^! \cong (\mathcal{O}(M_{m-k}^*)/\mathcal{O}(X))_m$$

induced by an explicit Fourier transform. (We note that if $m > n$, then $(\mathcal{O}(M_k)/\mathcal{O}(X))_m^! \cong (\mathcal{O}(M_k)/\mathcal{O}(X))_m$ by ambidexterity.)

All of the above dualities are, suitably interpreted, instances of this general duality; for example, the (connective) spectrum objects $\mathbb{Z}$ and $\mathbb{G}_m = \mathrm{Gest}_X(\mathbb{S}[t^{\pm 1}])$ over $X = \mathrm{Gest}(D(\mathbb{S}))$ are Cartier dual.

We will prove the theorem in the next lectures. Let us note the following corollary and examples now.

COROLLARY 10.9. *For any $M \in \bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$, the biduality map*

$$M \rightarrow (M^*)^*$$

is an isomorphism.

PROOF. Indeed, both sides live in $\bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$, and for any $N \in \bigcup_n \mathrm{Sp}(\mathrm{Gest}_X^{n\text{-ét}})$, we have

$$\mathrm{Hom}(N, (M^*)^*) = \mathrm{Hom}(N \otimes M^*, \mathrm{GL}_1) = \mathrm{Hom}(M^*, N^*)$$

identifying the natural map $\mathrm{Hom}(N, M) \rightarrow \mathrm{Hom}(N, (M^*)^*)$ with $\mathrm{Hom}(N, M) \rightarrow \mathrm{Hom}(M^*, N^*)$, the map obtained by contravariant functoriality of $M \mapsto M^*$. But this is an equivalence. $\square$

We note that this result imposes virtually no finiteness hypothesis on M ! This is in fact a bit unsettling. Namely, consider the following example.

EXAMPLE 10.10. Let us work over $X = \mathrm{Gest}(D(\mathbb{C}))$ (although $\mathbb{Z}[\frac{1}{p}, \zeta_p]$ would be sufficient), and fix a prime p . Then $\mathbb{Z}$ is Cartier dual to $\mathbb{G}_m = \mathrm{Gest}_X(\mathbb{C}[t^{\pm 1}])$. As the functor $\mathcal{H}\mathrm{om}(-, \mathrm{GL}_1)$ realizing Cartier duality is exact, it follows that $\mathbb{Z}/p$ is Cartier dual to

$$\mu_p = \ker(\mathbb{G}_m \xrightarrow{t \mapsto t^p} \mathbb{G}_m) \cong \mathbb{Z}/p,$$

by a choice of ζ_p . Moreover, on the derived category of sheaves of $\mathbb{F}_p$ -vector spaces, the duality $\mathcal{H}\mathrm{om}(-, \mathrm{GL}_1)$ becomes the naive duality $\mathcal{H}\mathrm{om}_{\mathbb{F}_p}(-, \mathbb{F}_p)$. Now, using biduality, we see that (with all sums and products taken in the derived category):

- (1) The 0-étale $\bigoplus_{\mathbb{N}} \mathbb{F}_p$ is dual to the 0-proper $\prod_{\mathbb{N}} \mathbb{F}_p$, which is hence 1-étale;
- (2) The 1-étale $\bigoplus_{\mathbb{N}} \prod_{\mathbb{N}} \mathbb{F}_p$ is dual to the 1-proper (hence 2-étale) $\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \mathbb{F}_p$;
- (3) The 2-étale $\bigoplus_{\mathbb{N}} \prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \mathbb{F}_p$ is dual to the 2-proper (hence 3-étale) $\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \prod_{\mathbb{N}} \mathbb{F}_p$; ...

It is, to me, extremely surprising that there is any topos which supports these bidualities. In naive vector spaces, already the first biduality for $\bigoplus_{\mathbb{N}} \mathbb{F}_p$ fails, but it can be repaired by passing to, for example, condensed $\mathbb{F}_p$ -vector spaces. In condensed $\mathbb{F}_p$ -vector spaces, it turns out that it is consistent that the derived biduality holds for $\bigoplus_{\mathbb{N}} \prod_{\mathbb{N}} \mathbb{F}_p$, cf. [BLH25], but already this is independent of ZFC (it fails if the continuum is “too small”). Also, it always fails if one replaces $\mathbb{N}$ by a larger cardinality, already for $\aleph_1$. And certainly (3) will not hold in condensed $\mathbb{F}_p$ -vector spaces.

Thus, putting a Gestalt structure on these vector space objects, one gets a surprisingly clean duality theory. We note that by our foundational choice of using $\aleph_1$ -presentable $(\infty, 1)$ -categories throughout, we need to use countable sums/products above; if we had chosen a larger cardinal there, larger sums/products would be allowed here as well. But I already find the dualities above unsettling, so am happy to ask them only for countable index sets.

EXAMPLE 10.11. Let us continue to work over $X = \text{Gest}(D(\mathbb{C}))$ (for the present discussion, the cyclotomic extension of $\mathbb{Q}$ would suffice). The spectrum $\text{GL}_1(X)$ is given by

$$\text{GL}_1(X) = (\mathbb{C}^\times, D(\mathbb{C})^\times, 1\text{Pr}_\mathbb{C}^\times, \dots).$$

It is in general hard to compute its homotopy groups, but certainly $\pi_0 = \mathbb{C}^\times$, $\pi_{-1} = \mathbb{Z}$ (as any $\otimes$ -invertible object in $D(\mathbb{C})$ is isomorphic to a shift $\mathbb{C}[n]$ for some n), and $\pi_{-2} = 0$ by a theorem of Stefanich [Ste25b]. Working with “small” higher linear categories, one also knows $\pi_{-3} = 0$ while π_{-4} is a highly nontrivial but known (countably generated, but not finitely generated) abelian group, known as the Witt group.

In the language used by mathematical physicists, these higher invertible linear categories in fact give rise to “invertible topological quantum field theories”, and there is an expectation (Kitaev, ...) that these are classified by the Brown–Comenetz dual of the sphere spectrum $I_{\mathbb{C}^\times}$ whose homotopy groups are

$$\pi_{-n} I_{\mathbb{C}^\times} = \text{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times).$$

More precisely, $I_{\mathbb{C}^\times}$ is characterized up to isomorphism by

$$\text{Hom}(\mathbb{Z}, I_{\mathbb{C}^\times}) = \mathbb{C}^\times;$$

the description of the negative homotopy groups turns out to follow from this. Roughly, the rationale behind this expectation is that an n -dimensional invertible topological quantum field theory should associate to a framed closed n -manifold an invertible number, i.e. an element of $\mathbb{C}^\times$, and moreover this association should be invariant under cobordism. But cobordism classes of (stably) framed closed n -manifolds are given by $\pi_n \mathbb{S}$ by the Thom isomorphism.

However, this expectation differs drastically from the actual answer above. But if we look not at the global sections of GL_1 , but rather at GL_1 as a sheaf on Gestalten, we get the expected answer:

COROLLARY 10.12. *The spectrum object GL_1 in $\text{Gest}/_{D(\mathbb{C})}$ is an internal Brown–Comenetz dual $I_{\mathbb{G}_m}$ in the sense that*

$$\mathcal{H}\text{om}(\mathbb{Z}, \text{GL}_1) = \mathbb{G}_m.$$

In particular, for all $n > 0$, one has

$$\pi_{-n} \text{GL}_1 \cong \text{Hom}(\pi_n \mathbb{S}, \pi_0 \mathbb{G}_m) = \text{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times)$$

in 0-truncated abelian groups in $\text{Gest}_{/D(\mathbb{C})}$, via the pairing

$$\pi_n \mathbb{S} \times \pi_{-n} \text{GL}_1 \rightarrow \pi_0 \mathbb{G}_m. \quad {}^{30}$$

Note that as $\pi_n \mathbb{S}$ is a finite abelian group, so is $\text{Hom}(\pi_n \mathbb{S}, \pi_0 \mathbb{G}_m) = \text{Hom}(\pi_n \mathbb{S}, \mathbb{C}^\times)$ (and it is abstractly isomorphic to $\pi_n \mathbb{S}$). Also note that this is generalizing our earlier discussion of Pic , with $\pi_0 \text{Pic} = \mu_2$.

PROOF. The first assertion is just coming from the duality between $\mathbb{Z}$ and $\mathbb{G}_m$. To deduce the description of the negative homotopy groups of GL_1 , note first that $\mathbb{G}_m$ is also the connective cover of GL_1 , and its 1-connective $\tau_{\geq 1} \mathbb{G}_m$ is rational (as for any $A \in \text{CAlg}(D(\mathbb{C}))$, one has $\tau_{>0} A^\times \cong \tau_{>0} A$ via the logarithm, and $\tau_{>0} A$ is rational). Thus, it follows that also

$$\mathcal{H}\text{om}(\mathbb{Z}, \tau_{\leq 0} \text{GL}_1) = \tau_{\leq 0} \mathbb{G}_m,$$

where now the right-hand side is concentrated in degree 0. Now the identification of the negative homotopy groups follows from general nonsense about Brown–Comenetz duality: If in any (countably) $(\infty, 1)$ -topos one has a coconnective spectrum object G such that $\mathcal{H}\text{om}(\mathbb{Z}, G) = A$ is concentrated in degree 0 and divisible, then $\pi_{-n} G \cong \text{Hom}(\pi_n \mathbb{S}, A)$ via the pairing $\pi_n \mathbb{S} \times \pi_{-n} G \rightarrow \pi_0 G = A$. $\square$

It is rather clear how most of the examples above fit into the setting of Gestalten. The most subtle one is Deligne’s example. For this, we need to use the transmutation operation from the last lecture, turning any six-functor formalism into a Gestalt. This could be done with the ℓ -adic six-functor formalism, but in fact one can give a more refined discussion by working in the motivic six-functor formalism SH. This last example is also meant to highlight the new ways in which one can do geometry with Gestalten.

EXAMPLE 10.13. Let $X = [\text{Spec}(\mathbb{Z})]_{\text{SH}} \in \text{Gest}$. Over it, we have the transmutation $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$ of the affine line, which has a natural structure as a commutative group (via addition); in fact, even a commutative ring structure, and in particular a module structure over itself. Thus, the Cartier dual

$$[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}^*$$

still carries the structure of an $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$ -module. If this was isomorphic to $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}$, we would get a Fourier-duality. This is not quite true, but it is locally true:

PROPOSITION 10.14. *There is a cover $\tilde{X} \rightarrow X$ such that*

$$[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}}^* \times_X \tilde{X} \cong ([\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}} \times_X \tilde{X})[-1]$$

as $[\mathbb{A}_{\mathbb{Z}}^1]_{\text{SH}} \times_X \tilde{X}$ -modules.

In fact, the universal example of such an $\tilde{X}$ will be a $[\mathbb{G}_{m, \mathbb{Z}}]_{\text{SH}}$ -torsor $\tilde{X} \rightarrow X$. Roughly, $\tilde{X}$ is given by freely adjoining an exponential local system on $\mathbb{A}_{\mathbb{Z}}^1$ to the motivic six-functor formalism; and the new six-functor formalism $S \mapsto \mathcal{O}([S]_{\text{SH}} \times_X \tilde{X})_1$ turns out to yield a theory of “exponential motives”, cf. [FJ25], and in particular [GPL22] which gives a direct construction of the six-functor formalism corresponding to $\tilde{X}$.

Over

$$[\text{Spec}(\mathbb{F}_p)]_{\text{SH}} \times_{\text{Gest}(\mathbb{S})} \text{Gest}(\mathbb{S}[\frac{1}{p}, \zeta_p]) \rightarrow X = [\text{Spec}(\mathbb{Z})]_{\text{SH}},$$

³⁰As GL_1 is 0-proper, everything is happening in the countably $(\infty, 1)$ -topos of 1-étale Gestalten.

we get an additive character ψ , and the Artin–Schreier cover yields an exponential local system $\mathcal{L}_\psi$ on $\mathbb{A}_{\mathbb{F}_p}^1$ that turns out to yield a splitting

$$[\mathrm{Spec}(\mathbb{F}_p)]_{\mathrm{SH}} \times_{\mathrm{Gest}(\mathbb{S})} \mathrm{Gest}(\mathbb{S}[\frac{1}{p}, \zeta_p]) \rightarrow \tilde{X}.$$

Thus, abbreviating $[\mathbb{A}_{\mathbb{F}_p}^1]_{\mathrm{SH}[\zeta_p]} := [\mathbb{A}_{\mathbb{F}_p}^1]_{\mathrm{SH}} \times_{\mathrm{Gest}(\mathbb{S})} \mathrm{Gest}(\mathbb{S}[\frac{1}{p}, \zeta_p])$, we get an isomorphism

$$[\mathbb{A}_{\mathbb{F}_p}^1]_{\mathrm{SH}[\zeta_p]}^* \cong [\mathbb{A}_{\mathbb{F}_p}^1]_{\mathrm{SH}[\zeta_p]}[-1],$$

yielding a motivic version of Deligne’s Fourier equivalence; and by base change (to the Gestalt corresponding to the ℓ -adic six-functor formalism), also gives Deligne’s Fourier equivalence.

We note that there is also a Betti realization

$$\mathrm{Gest}(D(\mathbb{S})) \rightarrow [\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}}$$

corresponding to the Ringgestalt $\mathbb{R}$ over $\mathrm{Gest}(D(\mathbb{S}))$; i.e. the pullback of the Ringgestalt $[\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}} \rightarrow X = [\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}}$ is $\mathbb{R} = \mathrm{Gest}(\mathrm{Shv}(\mathbb{R}, D(\mathbb{S})))$ over $\mathrm{Gest}(D(\mathbb{S}))$. Under this realization, the torsor $\tilde{X} \rightarrow X$ yields an $\mathbb{R}^\times$ -torsor over $\mathrm{Gest}(D(\mathbb{S}))$ which, up to a trivial $\{\pm 1\}$ -torsor, is the $\mathbb{R}_{>0}$ -torsor

$$\mathrm{Gest}(\mathbb{W}) \rightarrow \mathrm{Gest}(D(\mathbb{S}))$$

from [Sch25c] discussed previously.

11. Lecture XI: Proof of Cartier Duality, I

In this lecture, we start the proof of Cartier duality. Our goal in this lecture is to compute the internal Hom into GL_1 .

As a first step, let us note that GL_1 is actually representable, meaning that each functor

$$X \mapsto \mathcal{O}(X)_n^\times \in \mathrm{CGrp}$$

is representable. Here $\mathrm{CGrp} = \mathrm{CGrp}(\mathrm{Ani}) = D_{\geq 0}(\mathbb{S})$ is the $(\infty, 1)$ -category of $\mathbb{E}_\infty$ -groups in anima, which is equivalent to connective spectra. If $n = 0$, this is the functor

$$X \mapsto \mathcal{O}(X)_0^\times$$

which is representable by the 0-affine Gestalt $\mathrm{Gest}(\mathbb{S})$ where $\mathbb{S} \in \mathrm{CAlg}(\mathrm{Ani})$ is the free commutative group on one element. (In other words, after base change to $X = \mathrm{Gest}(A)$, this is representable by $\mathrm{Gest}_X(A_0[\mathbb{S}])$.) By shifting, it follows that for any n , the functor $X \mapsto \mathcal{O}(X)_n^\times$ is representable by the n -affine Gestalt

$$(\mathrm{GL}_1)_n = \mathrm{Gest}(\mathbb{S} \otimes (n-1)\mathrm{Pr})$$

where $\mathbb{S} \otimes (n-1)\mathrm{Pr} \in \mathrm{CAlg}(n\mathrm{Pr})$ is the free presentably symmetric monoidal (∞, n) -category on an invertible object; it is the image of $\mathbb{S} \in \mathrm{CAlg}(\mathrm{Ani})$ under the symmetric monoidal functor $\mathrm{Ani} \rightarrow n\mathrm{Pr}$, which sends 1 to $(n-1)\mathrm{Pr}$.

Each $(\mathrm{GL}_1)_n$ is the subgroup of invertible elements in a sheaf of commutative monoids

$$(\mathbb{A}^1)_n : X \mapsto \mathcal{O}(X)_n^{\simeq, \aleph_1} \in \mathrm{CMon} = \mathrm{CAlg}(\mathrm{Ani})$$

which is representable by

$$(\mathbb{A}^1)_n = \mathrm{Gest}(\mathrm{Fin}^\simeq \otimes (n-1)\mathrm{Pr})$$

where the groupoid of finite sets $\mathrm{Fin}^\simeq = \bigsqcup_{k \geq 0} */\Sigma_k \in \mathrm{CAlg}(\mathrm{Ani})$ is the free $\mathbb{E}_\infty$ -monoid in anima on one generator. (A priori, the appearance of the $\aleph_1$ -compactness condition in the functor of points of $(\mathbb{A}^1)_n$ may be confusing, but it is a consequence of our definitions of morphisms in $n\mathrm{Pr}$ as preserving $\aleph_1$ -compact objects. All $\otimes$ -invertible objects are $\aleph_1$ -compact, as the unit must always be $\aleph_1$ -compact.)

The first key computation is the following.

THEOREM 11.1. *Let $X = \mathrm{Gest}(A)$ be any Gestalt, and let $G \in \mathrm{CGrp}(\mathrm{Gest}/_X)$ be any commutative group in Gestalten over X , such that G is n -suave. Then*

$$\mathcal{H}\mathrm{om}_{\mathrm{CGrp}(\mathrm{Gest}/_X)}(G, (\mathrm{GL}_1)_n)$$

is representable by an n -affine Gestalt over X , namely

$$\mathrm{Gest}_X((\mathcal{O}(G)/A)_n^!),$$

where $(\mathcal{O}(G)/A)_n^! \in A_{n+1}$ has a natural commutative algebra structure constructed below.

In fact, more generally, if $G \in \mathrm{CMon}(\mathrm{Gest}/_X)$ is only a commutative monoid which is n -suave over X , then

$$\mathcal{H}\mathrm{om}_{\mathrm{CMon}(\mathrm{Gest}/_X)}(G, (\mathbb{A}^1)_n)$$

is representable by the n -affine Gestalt $\mathrm{Gest}_X((\mathcal{O}(G)/A)_n^!)$.³¹

³¹Thanks to Ben-Moshe for pointing out that the argument works in this generality. Note that if G is a group, then any monoid map to $(\mathbb{A}^1)_n$ factors over the maximal subgroup $(\mathrm{GL}_1)_n$.

Before giving the proof, let us first explain the commutative algebra structure on $(\mathcal{O}(G)/A)_n^!$, and discuss two examples.

To construct the commutative algebra structure, note that on the full subcategory $\text{Gest}_{/X}^{n\text{-suave}} \subset \text{Gest}_{/X}$, the functor

$$Y \mapsto (\mathcal{O}(Y)/A)_n^! \in A_{n+1}$$

is a covariant symmetric monoidal functor

$$\text{Gest}_{/X}^{n\text{-suave}} \rightarrow A_{n+1},$$

where the source is equipped with the cartesian symmetric monoidal structure. Indeed, being built from the left adjoint to pullbacks, it automatically acquires a colax symmetric monoidal structure, and the structure maps are isomorphisms by a Künneth formula for the left adjoint of pullback, for suave maps. Thus, the commutative monoid G in $\text{Gest}_{/X}^{n\text{-suave}}$ is sent to a commutative algebra in A_{n+1} , as desired. In particular, the unit of the commutative algebra $(\mathcal{O}(G)/A)_n^!$ is induced via covariant functoriality from the zero section $* \xrightarrow{0} G$, while the multiplication is induced from the addition map $G^2 \xrightarrow{+} G$.

EXAMPLE 11.2. Assume that $X = \text{Gest}(D(R))$ for some usual commutative ring.

- (1) Let $G = \mathbb{Z}$ (i.e., the disjoint union of $\mathbb{Z}$ many copies of X , with obvious commutative group structure). This is 0-étale, and in particular 0-suave. Moreover, its homology is given by $\bigoplus_{n \in \mathbb{Z}} R \in D(R)$, where the unit is given by the effect on homology of $* \xrightarrow{0} \mathbb{Z}$, and multiplication induced by the effect on homology of $\mathbb{Z}^2 \xrightarrow{+} \mathbb{Z}$. This is just the algebra $R[t^{\pm 1}]$ of Laurent polynomials. Thus, the theorem says that

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}_{/X})}(\mathbb{Z}, \text{GL}_1) = \text{Gest}_X(R[t^{\pm 1}]) = \mathbb{G}_m.$$

Note that this is not yet proving that $\mathbb{Z}$ and $\mathbb{G}_m$ are Cartier dual: A priori the internal Hom in spectrum objects might have some nonconnective part.

- (2) Let G be a finite flat group scheme over R . Recall that finite flat group schemes are local complete intersections (for example, as by a theorem of Raynaud they can Zariski locally be embedded in abelian varieties, with quotient again an abelian variety, so that they are the kernel of a surjective map between smooth group schemes), and finite local complete intersections are 0-suave. Thus, we can again take $n = 0$. Moreover, the homology $(\mathcal{O}(G)/A)_0^!$ of G is the dual $\mathcal{O}(G)^*$ of the cohomology of G , as $\mathcal{O}(G)$ is finite projective and hence reflexive. The commutative algebra structure is then given by Cartier's construction. Thus, the theorem says

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}_{/X})}(G, \text{GL}_1) = G^*.$$

Again, a priori the internal Hom in spectrum objects might have a nonconnective part (but will turn out not to).

It turns out that the definition and basic properties of n -suave maps make Theorem 11.1 almost tautological:

PROOF OF THEOREM 11.1. First, let H be any n -suave Gestalt over X . Then, for any Gestalt Y over X , we can compute

$$\mathcal{H}\text{om}_{\text{Gest}_{/Y}}(H \times_X Y, (\mathbb{A}^1)_n|_Y).$$

Indeed, by definition of the functor of points of $(\mathbb{A}^1)_n$, this is given by the anima of $\aleph_1$ -compact objects in

$$\mathcal{O}(H \times_X Y)_n.$$

But as $H \times_X Y \rightarrow Y$ is n -suave, this is also given by

$$\mathrm{Hom}_{\mathcal{O}(Y)_{n+1}}((\mathcal{O}(H \times_X Y)/\mathcal{O}(Y))_n^!, 1).$$

Moreover, the formation of $(\mathcal{O}(-)/\mathcal{O}(Y))_n^!$ commutes with base change for n -suave maps, so $(\mathcal{O}(H \times_X Y)/\mathcal{O}(Y))_n^!$ is the image of $(\mathcal{O}(H)/A)_n^!$ under $A_{n+1} = \mathcal{O}(X)_{n+1} \rightarrow \mathcal{O}(Y)_{n+1}$. By adjunction, we can then rewrite the last display as

$$\mathrm{Hom}_{\mathcal{O}(X)_{n+1}}((\mathcal{O}(H)/A)_n^!, (\mathcal{O}(Y)/\mathcal{O}(X))_n).$$

Now if $H = G$ is a commutative monoid, then we can chase the commutative monoid structure through these identifications, and we see that maps as commutative monoids

$$\mathcal{H}\mathrm{om}_{\mathrm{CMon}(\mathrm{Gest}_Y)}(G \times_X Y, (\mathbb{A}^1)_n|_Y)$$

are given by

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathcal{O}(X)_{n+1})}((\mathcal{O}(G)/A)_n^!, (\mathcal{O}(Y)/\mathcal{O}(X))_n).$$

But the latter is just maps from Y to the n -affine Gestalt $\mathrm{Gest}_X((\mathcal{O}(G)/A)_n^!)$ over X , giving the desired result. $\square$

A consequence of Theorem 11.1 is the following representability result for the Cartier dual. We state it under the minimal hypothesis required for the proof.³²

COROLLARY 11.3. *If $G \in \mathrm{CGrp}(\mathrm{Gest}_X)$ is eventually suave, then for any n , the object*

$$\mathcal{H}\mathrm{om}_{\mathrm{CGrp}(\mathrm{Gest}_X)}(G, (\mathrm{GL}_1)_n)$$

is representable in $\mathrm{CGrp}(\mathrm{Gest}_X)$.

Similarly, if

$$M = (M_0, M_1, \dots) \in \mathrm{Sp}(\mathrm{Gest}_X) = \lim_{\Omega} \mathrm{CGrp}(\mathrm{Gest}_X)$$

is a spectrum object in Gestalten over X such that M_n is eventually suave for infinitely many n , then

$$\mathcal{H}\mathrm{om}_{\mathrm{Sp}(\mathrm{Gest}_X)}(M, \mathrm{GL}_1)$$

is representable in $\mathrm{Sp}(\mathrm{Gest}_X)$.

PROOF. For the first part, note that if G is m -suave, then we already know the representability for $n = m$; but as m -suave implies m' -suave for $m' \geq m$, this gives representability for $n \geq m$. On the other hand, as $(\mathrm{GL}_1)_{n-1} = \Omega(\mathrm{GL}_1)_n$, we have

$$\mathcal{H}\mathrm{om}_{\mathrm{CGrp}(\mathrm{Gest}_X)}(G, (\mathrm{GL}_1)_{n-1}) = \Omega \mathcal{H}\mathrm{om}_{\mathrm{CGrp}(\mathrm{Gest}_X)}(G, (\mathrm{GL}_1)_n),$$

and hence representability for some n implies it for $n - 1$, and then inductively all smaller values.

If $M = (M_0, M_1, \dots) \in \mathrm{Sp}(\mathrm{Gest}_X)$, then

$$\tau_{\geq 0} \mathcal{H}\mathrm{om}_{\mathrm{Sp}(\mathrm{Gest}_X)}(M, \mathrm{GL}_1) = \lim_n \mathcal{H}\mathrm{om}_{\mathrm{CGrp}(\mathrm{Gest}_X)}(M_n, (\mathrm{GL}_1)_n).$$

³²If all objects involved are n -étale for some fixed n , this representability is essentially a tautology by using that we are essentially working in an $(\infty, 1)$ -topos.

We can restrict to any cofinal subset of indices. By assumption, there is a cofinal system where M_n is eventually suave, for which the internal Hom is representable; as limits of representable objects are still representable, this gives the result for $\tau_{\geq 0} \mathcal{H}om_{\mathrm{Sp}(\mathrm{Gest}/X)}(M, \mathrm{GL}_1)$. But by shifting, all other truncations are representable as well. $\square$

For the proof of Cartier duality, we need to compare homomorphisms in commutative groups with the homomorphisms in spectrum objects. The key result is the following slightly technical result. In its formulation, we use the concept of n -affine truncation, which for a morphism $f : Y \rightarrow X$ of Gestalten refers to the universal n -affine map over X to which Y maps; this is given by

$$n\mathrm{Aff}_X(Y) = \mathrm{Gest}_X((\mathcal{O}(Y)/\mathcal{O}(X))_n) \rightarrow X.$$

Note that $Y = \lim_n n\mathrm{Aff}_X(Y)$ is the limit of its n -affine truncations (as the corresponding colimit of Stefanich- $\mathcal{O}(X)$ -algebras stabilizes in each degree). Beware, however, that in general the formation of n -affine truncations does not commute with base change; it does if f is n -prim.

Also note that in the following theorem, the assumption that $* \rightarrow M_{k+1}$ is k -suave implies, via base change along $* \rightarrow M_{k+1}$, that $M_k = \Omega M_{k+1} \rightarrow *$ is k -suave; in general the converse is not true (but by descent, it holds if $* \rightarrow M_{k+1}$ is surjective, i.e. if $M_{k+1} = */M_k$).

THEOREM 11.4. *Let $M = (M_0, M_1, \dots) \in \mathrm{Sp}(\mathrm{Gest}/X)$ be a spectrum object in Gestalten over X . Assume that for all $k \geq n$, the map*

$$* \rightarrow M_{k+1}$$

is k -suave. Let $M^ = (M_0^*, M_1^*, \dots)$ be the Cartier dual, which we already know to be representable. Then M_0^* is n -prim. Moreover, the natural map*

$$M_0^* \rightarrow \mathcal{H}om_{\mathrm{CGrp}(\mathrm{Gest}/X)}(M_n, (\mathrm{GL}_1)_n)$$

induces an isomorphism

$$n\mathrm{Aff}_X(M_0^*) \cong \mathcal{H}om_{\mathrm{CGrp}(\mathrm{Gest}/X)}(M_n, (\mathrm{GL}_1)_n).$$

REMARK 11.5. Note that by Theorem 11.1, the right-hand side

$$\mathcal{H}om_{\mathrm{CGrp}(\mathrm{Gest}/X)}(M_n, (\mathrm{GL}_1)_n)$$

is n -affine, and explicit, corresponding to the commutative algebra $(\mathcal{O}(M_n)/A)_n^!$ (where $A = \mathcal{O}(X)$ as before). Thus, Theorem 11.4 yields an isomorphism

$$(\mathcal{O}(M_0^*)/A)_n \cong (\mathcal{O}(M_n)/A)_n^!.$$

Also note that the theorem also applies with n replaced by any $m \geq n$, so we also get

$$(\mathcal{O}(M_0^*)/A)_m \cong (\mathcal{O}(M_m)/A)_m^!$$

for any $m \geq n$. One can also apply the theorem to shifts of M , which yields

$$(\mathcal{O}(M_k^*)/A)_{m+k} \cong (\mathcal{O}(M_m)/A)_{m+k}^!$$

for $k \geq 0, m \geq n$.

Before starting the proof, we prove the key statement as a separate lemma.

LEMMA 11.6. *Let $G \in \text{CGrp}(\text{Gest}/_X)$ and assume that G is $n+1$ -suave, while $* \rightarrow G$ is n -suave. Let $G' = \Omega G \in \text{CGrp}(\text{Gest}/_X)$; note that G' is n -suave. Then $\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G, (\text{GL}_1)_{n+1})$ is n -prim, and the natural map*

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G, (\text{GL}_1)_{n+1}) \xrightarrow{\Omega} \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G', (\text{GL}_1)_n)$$

induces an isomorphism

$$n\text{Aff}_X(\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G, (\text{GL}_1)_{n+1})) \cong \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G', (\text{GL}_1)_n).$$

PROOF. By Theorem 11.1,

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G, (\text{GL}_1)_{n+1}) = \text{Gest}_X((\mathcal{O}(G)/A)_{n+1}^!)$$

is $n+1$ -affine, while

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/_X)}(G', (\text{GL}_1)_n) = \text{Gest}_X((\mathcal{O}(G')/A)_n^!)$$

is n -affine. We need to see that the unit map $1 \rightarrow (\mathcal{O}(G)/A)_{n+1}^!$ in A_{n+2} admits a right adjoint, and the map

$$(\mathcal{O}(G')/A)_n^! \rightarrow \text{End}_{(\mathcal{O}(G)/A)_{n+1}^!}(1)$$

is an isomorphism in A_{n+1} . Recall that the unit map

$$1 \rightarrow (\mathcal{O}(G)/A)_{n+1}^!$$

in A_{n+2} is given by the effect on homology of $i : * \xrightarrow{0} G$. More precisely, this effect on homology already gives a map

$$i_{n,!} : (\mathcal{O}(*)/\mathcal{O}(G))_{n+1}^! \rightarrow 1$$

in $\mathcal{O}(G)_{n+2}$, and then we consider

$$f_{n+1,\sharp}(i_{n,!}) : 1 \rightarrow (\mathcal{O}(G)/A)_{n+1}^!$$

where $f : G \rightarrow * = X$ is the projection. As i is n -suave, the map $i_n^!$ admits a right adjoint $i_n^!$, and hence also $f_{n+1,\sharp}(i_{n,!})$ admits the right adjoint $f_{n+1,\sharp}(i_n^!)$, giving n -primness. Moreover, the composite map

$$1 \xrightarrow{f_{n+1,\sharp}(i_{n,!})} (\mathcal{O}(G)/A)_{n+1}^! \xrightarrow{f_{n+1,\sharp}(i_n^!)} 1,$$

of $f_{n+1,\sharp}(i_{n,!})$ with its internal right adjoint $f_{n+1,\sharp}(i_n^!)$, is the object of $A_{n+1} = \text{End}_{A_{n+2}}(1)$ computing $\text{End}_{(\mathcal{O}(G)/A)_{n+1}^!}(1)$. Taking the internal Hom to 1 of the displayed diagram does not change the composite, and yields the composite

$$1 \xrightarrow{f_{n+1,*}(i_{n,\sharp}^*)} (\mathcal{O}(G)/A)_{n+1} \xrightarrow{f_{n+1,*}(i_n^*)} 1.$$

Thus, we need to compute $f_{n+1,*}(i_{n,\sharp}^*)$. As $f_{n+1,*} : \mathcal{O}(G)_{n+2} \rightarrow A_{n+2}$ is just a forgetful functor, we drop it from the notation. Using base change in the cartesian diagram

$$\begin{array}{ccc} G' & \xrightarrow{g} & * \\ g \downarrow & & \downarrow i \\ * & \xrightarrow{i} & G, \end{array}$$

we see that $i_n^* i_{n,\sharp} = g_{n,\sharp} g_n^*$, and the map

$$1 \xrightarrow{g_{n,\sharp} g_n^*} 1$$

in A_{n+2} corresponds to

$$g_{n,\sharp} 1 = (\mathcal{O}(Y)/A)_n! \in A_{n+1} = \text{End}_{A_{n+2}}(1). \quad \square$$

Now we can deduce Theorem 11.4.

PROOF OF THEOREM 11.4. First, we show that $\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_m, (\text{GL}_1)_m)$ is n -prim over X for all $m \geq n$, and the map

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_m, (\text{GL}_1)_m) \xrightarrow{\Omega^{m-n}} \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_n, (\text{GL}_1)_n)$$

induces an isomorphism

$$n\text{Aff}_X(\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_m, (\text{GL}_1)_m)) \cong \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_n, (\text{GL}_1)_n).$$

We induct on m . For $m = n + 1$, this is Lemma 11.6. In general, this also shows that

$$\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_{m+1}, (\text{GL}_1)_{m+1})$$

is m -prim, and taking its m -affine truncation yields $\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_m, (\text{GL}_1)_m)$. But being m -prim, with n -prim m -affine truncation, implies being n -prim; and if we also take n -affine truncation, we get $\mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_n, (\text{GL}_1)_n)$ by induction.

Now

$$\tau_{\geq 0} \mathcal{H}\text{om}_{\text{Sp}(\text{Gest}/X)}(M, \text{GL}_1) = \lim_m \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_m, (\text{GL}_1)_m),$$

and countable limits of n -prim maps are n -prim, and n -affine truncation commutes with this countable limit. Thus, we also have

$$n\text{Aff}_X(\tau_{\geq 0} \mathcal{H}\text{om}_{\text{Sp}(\text{Gest}/X)}(M, \text{GL}_1)) = \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}/X)}(M_n, (\text{GL}_1)_n). \quad \square$$

12. Lecture XII: Proof of Cartier Duality, II

In this lecture, we finish the proof of Cartier Duality. We start with a refinement of Theorem 11.4.

PROPOSITION 12.1. *In Theorem 11.4, the map*

$$* \rightarrow M_{k+1}^*$$

is $n + k$ -prim for $k \geq 0$.

We note that it already follows from Theorem 11.4 and shifting that $M_k^* \rightarrow *$ is $n + k$ -prim for $k \geq 0$. The present statement is slightly stronger.

PROOF. It suffices to show that $i : * \rightarrow M_1^*$ is n -prim (as the assertion for other values of k follows by applying the result for different values of n). In order to reduce the number of variables further, we note that by shifting the Stefanich ring A , we can moreover assume $n = 0$.

Note that for $m \geq 1$ the map

$$i_{m-1}^* : (\mathcal{O}(M_1^*)/A)_m \rightarrow 1$$

in A_{m+1} is identified with the map

$$f_{m-1,!} : (\mathcal{O}(M_{m-1})/A)_m^! \rightarrow 1$$

where $f : M_{m-1} \rightarrow * = X$ is the projection. Indeed, this follows from the functoriality (under the map $M \rightarrow M' = 0$) of the identification of $(\mathcal{O}(M_1^*)/A)_m$ with $(\mathcal{O}(M_{m-1})/A)_m^!$. As $M_{m-1} \rightarrow *$ is $(m-1)$ -suave, the map $f_{m-1,!}$ has a right adjoint in A_{m+1} ; thus, i_{m-1}^* has a right adjoint

$$i_{m-1,*} : 1 \rightarrow (\mathcal{O}(M_1^*)/A)_m$$

in A_{m+1} , for all $m \geq 1$. This is however not yet the statement we need for showing that i is 0-prim: We need a right adjoint already in $\mathcal{O}(M_1^*)_{m+1}$. We can get at this by looking at the unit map

$$\text{id} \rightarrow i_{m,*} i_m^*$$

in $\text{End}_{A_{m+2}}((\mathcal{O}(M_1^*)/A)_{m+1})$. Evaluating this map at the unit, i.e. restricting along $1 \rightarrow \mathcal{O}(M_1^*)_{m+1}$, we get the map

$$1 \rightarrow i_{m,*} 1$$

in $\text{Hom}_{A_{m+2}}(1, (\mathcal{O}(M_1^*)/A)_{m+1}) = \mathcal{O}(M_1^*)_{m+1}$; this is the map that we want to see has a right adjoint. For this, it suffices that the unit map $\text{id} \rightarrow i_{m,*} i_m^*$ has a right adjoint. Translating, this means that we want the unit map

$$\text{id} \rightarrow f_m^! f_{m,!} \text{ in } \text{End}_{A_{m+2}}((\mathcal{O}(M_m)/A)_{m+1}^!)$$

to have right adjoint, where now f denotes the map $M_m \rightarrow *$. But there is a functor

$$\mathcal{O}(M_m \times M_m)_{m+1} \rightarrow \text{End}_{A_{m+2}}((\mathcal{O}(M_m)/A)_{m+1}^!)$$

taking $K \in \mathcal{O}(M_m \times M_m)_{m+1}$ to the map

$$(\mathcal{O}(M_m)/A)_{m+1}^! \xrightarrow{p_m^!} (\mathcal{O}(M_m \times M_m)/A)_{m+1}^! \xrightarrow{-\otimes K} (\mathcal{O}(M_m \times M_m)/A)_{m+1}^! \xrightarrow{q_{m,!}} (\mathcal{O}(M_m)/A)_{m+1}^!$$

where $p, q : M_m \times M_m \rightarrow M_m$ are the two projections, both of which are m -suave. (To define $-\otimes K$, note that $(\mathcal{O}(M_m \times M_m)/A)_{m+1}^!$ is the image of the unit under the left adjoint $\mathcal{O}(M_m \times M_m)_{m+2} \rightarrow$

A_{m+2} to pullback; and hence the endomorphisms $\mathcal{O}(M_m \times M_m)_{m+1} = \text{End}_{\mathcal{O}(M_m \times M_m)_{m+2}}(1)$ act.) Moreover, the map

$$\Delta_{f,m-1,!} : (\mathcal{O}(M_m)/\mathcal{O}(M_m \times M_m))_m^! \rightarrow 1$$

in $\mathcal{O}(M_m \times M_m)_{m+1}$ realizes to the map $\text{id} \rightarrow f_m^! f_{m,!}$. Thus, it suffices to see that $\Delta_{f,m-1,!}$ has a right adjoint in $\mathcal{O}(M_m \times M_m)_{m+1}$. But this follows from Δ_f being $(m-1)$ -suave; note that $\Delta_f : M_m \rightarrow M_m \times M_m$ is a pullback of $* \rightarrow M_m$. $\square$

COROLLARY 12.2. *Consider the subcategory $\text{Sp}(\text{Gest}_X)^{\text{nice}}$ of all $M \in \text{Sp}(\text{Gest}_X)$ such that for some n , the map $* \rightarrow M_k$ is $(n+k)$ -suave for all $k \geq n$. For any such M , the Cartier dual M^* is representable by an object of $\text{Sp}(\text{Gest}_X)^{\text{nice}}$, and the biduality map*

$$M \rightarrow (M^*)^*$$

is an isomorphism. Thus, $M \mapsto M^$ induces an anti-self-equivalence of $\text{Sp}(\text{Gest}_X)^{\text{nice}}$.*

PROOF. For any $M \in \text{Sp}(\text{Gest}_X)^{\text{nice}}$, a suitable shift of M satisfies the hypotheses of Theorem 11.4. By Proposition 12.1, the dual M^* has the property that $* \rightarrow M_{k+1}^*$ is $(n+k)$ -prim for $k \geq 0$. By Proposition 6.21, this implies that $* \rightarrow M_{k+1}^*$ is $(n+k+1)$ -suave for $k \geq 1$. Thus, M^* still lies in $\text{Sp}(\text{Gest}_X)^{\text{nice}}$.

Moreover, if $* \rightarrow M_k$ is $n+k$ -suave for all $k \geq n$, the object $(\mathcal{O}(M_k^*)/A)_m^!$ is dualizable (and selfdual) for $k \geq n+1$ and $m \geq n+k+1$ by Proposition 6.21 and Lemma 6.22. In this regime, the identification

$$(\mathcal{O}(M_k^*)/A)_m \cong (\mathcal{O}(M_{m-k})/A)_m^!$$

is induced by the natural invertible object in

$$(\mathcal{O}(M_k^*)/A)_m \otimes (\mathcal{O}(M_{m-k})/A)_m \cong (\mathcal{O}(M_k^* \times M_{m-k})/A)_m$$

coming from the map

$$M_{m-k} \otimes M_k^* \rightarrow (\text{GL}_1)_m.$$

This factors over the pairing

$$(M^*)_{m-k}^* \otimes M_k^* \rightarrow (\text{GL}_1)_m,$$

which similarly induces an object in

$$(\mathcal{O}(M_k^*)/A)_m \otimes (\mathcal{O}((M^*)_{m-k}^*)/A)_m$$

yielding a perfect duality, a priori for $k \geq n+2$ and $m \geq n+k+2$. It follows that the map

$$(\mathcal{O}(M_{m-k})/A)_m \rightarrow (\mathcal{O}((M^*)_{m-k}^*)/A)_m$$

must be an equivalence, as both are dual to $(\mathcal{O}(M_k^*)/A)_m$ via compatible pairings. As this holds for all $m \geq n+k+2$, the map $M_k \rightarrow (M^*)_k^*$ is an isomorphism; and as this holds for all $k \geq n+2$, the map $M \rightarrow (M^*)^*$ is an isomorphism. $\square$

The previous corollary already gives us a perfect Cartier selfduality, on an even larger class of spectrum objects than promised in Theorem 10.8. Under this equivalence, one can translate various properties.

THEOREM 12.3. *Let $M \in \text{Sp}(\text{Gest}_X)^{\text{nice}}$, with Cartier dual M^* , and let $n \geq 0$.*

- (i) For all $k \geq n$, the map $* \rightarrow M_{k+1}$ is k -suave, if and only if for all $k \geq 0$, the map $* \rightarrow M_{k+1}^*$ is $n+k$ -prim.³³
- (ii) For all $k \in \mathbb{Z}$, the map $M_k \rightarrow *$ is n -étale, if and if for all $k \in \mathbb{Z}$, the map $M_k^* \rightarrow *$ is n -proper.
- (iii) Assume that $* \rightarrow M_{k+1}$ is k -prim for all $k \geq n$. Then $M_{n+1} \rightarrow *$ is $n+1$ -affine if and only if M^* is connective.

Thus, informally, Cartier duality exchanges suave and prim, étale and proper, and affine and connective. The first two dualities are at least morally unsurprising, whereas this last duality, between affineness and connectivity, is a new phenomenon. Note that it follows from Proposition 8.1 that in (iii), the condition “ $M_{n+k} \rightarrow *$ is $n+k$ -affine” is independent of $k \geq 1$, and is also equivalent to $* \rightarrow M_{n+1}$ being n -affine.

PROOF. Note that the condition in (ii) is just condition (i) applied to all shifts of M ; thus (i) implies (ii). We have already seen the forward direction of (i) in Proposition 12.1. The proof of the converse is mostly a direct translation of the proof of Proposition 12.1. Again, it is enough to show that $i : * \rightarrow M_1$ is n -suave, and we can assume for simplicity that $n = 0$ by shifting the Stefanich ring A . We note that the assumption that $* \rightarrow M_{k+1}^*$ is k -prim for all $k \geq 0$ implies that $* \rightarrow M_{k+1}^*$ is $k+1$ -suave for all $k \geq 1$ by ambidexterity, and hence by using the forward direction, we already see that $* \rightarrow M_{k+1}$ is $k+1$ -prim for all $k \geq 1$. In particular, by base change, we see that $M_k \rightarrow *$ is $k+1$ -prim for all $k \geq 1$. Moreover, the forward direction gives us, for $m \geq 1$, an identification of the map

$$i_m^* : (\mathcal{O}(M_1)/A)_{m+1} \rightarrow 1$$

in A_{m+2} with the map

$$f_{m,!} : (\mathcal{O}(M_m^*)/A)_{m+1}^! \rightarrow 1$$

where $f : M_m^* \rightarrow * = X$ is the projection. Note that as f and its diagonal are m -prim, it follows from ambidexterity that $(\mathcal{O}(M_m^*)/A)_{m+1}^!$ is dualizable, and the map $f_{m,!}$ is dual to the map

$$f_m^* : 1 \rightarrow (\mathcal{O}(M_m^*)/A)_{m+1}.$$

As f is m -prim, this map admits a right adjoint; reversing the identifications then shows that i_m^* has a left adjoint

$$i_{m,\sharp} : 1 \rightarrow (\mathcal{O}(M_1)/A)_{m+1}$$

in A_{m+2} , for $m \geq 1$. Again, this information is too weak — we need this internally in $\mathcal{O}(M_1)_{m+2}$, not just A_{m+2} , but even more importantly we need to get down to $m \geq 0$. But note that existence of $i_{m,\sharp}$ in A_{m+2} already implies it after further forgetting down to $(m+1)\text{Pr}$, and in particular the pullback functor

$$i_m^* : \mathcal{O}(M_1)_{m+1} \rightarrow A_{m+1}$$

has a left adjoint still denoted $i_{m,\sharp}$ in $(m+1)\text{Pr}$, for all $m \geq 1$. In particular, we can define $(\mathcal{O}(*)/\mathcal{O}(M_1))_m^! \in \mathcal{O}(M_1)_{m+1}$ for all $m \geq 1$, and suaveness of i is equivalent to the assertion that the map

$$i_{m-1,!} : (\mathcal{O}(*)/\mathcal{O}(M_1))_m^! \rightarrow 1$$

in $\mathcal{O}(M_1)_{m+1}$ has a right adjoint, for all $m \geq 1$.

³³That a “suave-prim-duality” of roughly this shape — asking it not for $M_0 \rightarrow *$ but $* \rightarrow M_1$ — should be true is something I learned from Johannes Anschütz and Arthur-César le Bras.

The idea is now again to look at the induced counit map

$$i_{m,\sharp} i_m^* \rightarrow \text{id}$$

in $\text{End}_{A_{m+2}}((\mathcal{O}(M_1)/A)_{m+1})$. Restricting along the unit $1 \rightarrow (\mathcal{O}(M_1)/A)_{m+1}$, this gives the map

$$i_{m-1,!} : (\mathcal{O}(M_1)/A)_m^! = i_{m,\sharp} 1 \rightarrow 1.$$

Thus, it suffices to see that this counit map has a right adjoint. Under the duality between $f_{m,!}$ and f_m^* , this means that the counit map

$$f_m^* f_{m,*} \rightarrow \text{id} \text{ in } \text{End}_{A_{m+2}}((\mathcal{O}(M_m^*)/A)_{m+1})$$

has a right adjoint. But there is a functor

$$\mathcal{O}(M_m \times M_m)_{m+1} \rightarrow \text{End}_{A_{m+2}}((\mathcal{O}(M_m)/A)_{m+1})$$

taking $K \in \mathcal{O}(M_m \times M_m)_{m+1}$ to the map

$$(\mathcal{O}(M_m)/A)_{m+1} \xrightarrow{p_m^*} (\mathcal{O}(M_m \times M_m)/A)_{m+1} \xrightarrow{-\otimes K} (\mathcal{O}(M_m \times M_m)/A)_{m+1} \xrightarrow{q_{m,*}} (\mathcal{O}(M_m)/A)_{m+1}$$

where $p, q : M_m \times M_m \rightarrow M_m$ are the two projections, both of which are m -prim. Moreover, the map

$$\Delta_{f,m-1}^* : 1 \rightarrow (\mathcal{O}(M_m)/\mathcal{O}(M_m \times M_m))_m$$

in $\mathcal{O}(M_m \times M_m)_{m+1}$ realizes to the map $f_m^* f_{m,*} \rightarrow \text{id}$. Thus, it suffices to see that $\Delta_{f,m-1}^*$ has a right adjoint in $\mathcal{O}(M_m \times M_m)_{m+1}$. But this follows from Δ_f being $(m-1)$ -prim; note that $\Delta_f : M_m \rightarrow M_m \times M_m$ is a pullback of $* \rightarrow M_m$.

For part (iii), we can again assume $n = 0$, by shifting the Stefanich ring A . We see from (i) that the hypothesis implies that M^* has the property that $* \rightarrow M_{k+1}^*$ is k -suave for all $k \geq 0$. We can then apply Theorem 11.4 to M^* and its shifts; this shows (using biduality) that

$$(n+k)\text{Aff}_X(M_k) \cong \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}_X)}(M_n^*, (\text{GL}_1)_{n+k})$$

for all $n, k \geq 0$. Now M^* is connective if and only if the right-hand side is independent of n , for all k ; this translates into the statement that $(n+k)\text{Aff}_X(M_k)$ is independent of n , which means that M_k is k -affine for all $k \geq 0$. But by Proposition 8.1, this is equivalent (for $k \geq 1$) to the condition that $* \rightarrow M_k$ is $k-1$ -affine, and these conditions are all equivalent, by another application of Proposition 8.1, and imply that $M_0 \rightarrow *$ is 0-affine. $\square$

Together with Remark 11.5, this basically finishes the proof of Theorem 10.8. There is one more thing to do, namely to show that all the examples fit into this general framework. In Theorem 11.1, we made some partial progress on this, but could not yet control the nonconnective Cartier dual. Using part (iii) of Theorem 12.3, we can improve on this.

PROPOSITION 12.4. *Let $G \in \text{CGrp}(\text{Gest}_X)$ and assume that G is n -suave, and $BG = */G$ is $n+1$ -affine. Then the Cartier dual $G^* \in \text{Sp}(\text{Gest}_X)$ of G is $-n$ -connective, and determined by*

$$G_n^* = \mathcal{H}\text{om}_{\text{CGrp}(\text{Gest}_X)}(G, (\text{GL}_1)_n) = \text{Gest}((\mathcal{O}(G)/A)_n^!).$$

PROOF. The commutative group G gives rise to the spectrum object $M = (G, BG, B^2G, \dots)$ where $BG = */G$ is the classifying space of G , $B^2G = */BG$ is the classifying space of BG , etc. . Then each $M_k = B^k G$ is a colimit of products of copies of G , and hence n -suave; even the maps $* \rightarrow B^k G$ are n -suave for $k \geq 1$. By ambidexterity, it follows that $B^k G \rightarrow *$ is $n+1$ -prim for $k \geq 1$,

and then the assumption that $BG \rightarrow *$ is $n + 1$ -affine implies that $B^k G$ is $n + k$ -affine for all $k \geq 1$ (in fact, the condition “ $B^k G$ is $n + k$ -affine” is independent of $k \geq 1$, and is implied by G being n -affine). Thus, Theorem 12.3 shows that M^* is $-n$ -connective, and then Theorem 11.1 computes M_n^* . $\square$

REMARK 12.5. To show that BG is $n + 1$ -affine, it suffices to show that G is $n + 1$ -affine, and the map

$$1 \rightarrow (\mathcal{O}(G)/A)_{n+1}$$

with its left adjoint (which exists as G is n -suave) yield a comonadic adjunction in A_{n+2} .³⁴ Indeed, BG will always be $n + 1$ -prim by ambidexterity, hence surject onto its $n + 1$ -affine truncation by Corollary 8.14; to show that this map is an isomorphism, it suffices to see that the map $G \rightarrow * \times_{(n+1)\mathrm{Aff}_X(BG)} *$ is an isomorphism. As everything in sight is $n + 1$ -affine, this amounts to showing that the map

$$1 \otimes_{(\mathcal{O}(BG)/A)_{n+1}} 1 \rightarrow (\mathcal{O}(G)/A)_{n+1}$$

in A_{n+2} is an isomorphism. But the comonadic adjunction $1 \rightarrow (\mathcal{O}(G)/A)_{n+1}$ in A_{n+2} descends to a similar comonadic adjunction in $\mathcal{O}(BG)_{n+2}$, and hence also (by forgetting structure) in A_{n+2} , which now reads

$$(\mathcal{O}(BG)/A)_{n+1} \rightarrow 1.$$

Base-changing this comonadic adjunction along $(\mathcal{O}(BG)/A)_{n+1} \rightarrow 1$ then gives back the original comonadic adjunction $1 \rightarrow (\mathcal{O}(G)/A)_{n+1}$.

It is now a simple matter to apply the proposition to compute the Cartier duals in all examples mentioned; one always gets the expected answer. Let us finish this lecture by discussing the motivic Fourier transform, i.e. prove Proposition 10.14. We repeat the statement.

PROPOSITION 12.6. *Let $X = [\mathrm{Spec}(\mathbb{Z})]_{\mathrm{SH}}$ be the transmutation of the Morel–Voevodsky motivic 6-functor formalism SH . There is a cover $\tilde{X} \rightarrow X$ such that*

$$[\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}}^* \times_X \tilde{X} \cong ([\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}} \times_X \tilde{X})[-1]$$

as $[\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}} \times_X \tilde{X}$ -modules.

PROOF. We apply Proposition 12.4 to $G = [\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}}$ and $n = 1$; note that G is 1-affine, hence in particular BG is 2-affine. It follows that M^* is (-1) -connective, and determined by

$$M_1^* = \mathrm{Gest}_X((\mathcal{O}(G)/A)_1^!).$$

In a six-functor formalism, the left and right adjoints to pullback agree from the second categorical level, so $(\mathcal{O}(G)/A)_1^! = (\mathcal{O}(G)/A)_1$, and is just the object $[\mathbb{A}_{\mathbb{Z}}^1] \in K_{\mathrm{SH}}$ in the presentable $(\infty, 2)$ -category of kernels. Moreover, the symmetric monoidal structure is the convolution symmetric monoidal structure.

We note that the connective cover of M^* is trivial — this follows by applying Theorem 11.1 to G and $n = 0$. This computes the connective cover of M^* as $\mathrm{Gest}_X((\mathcal{O}(G)/A)_0^!)$. But $(\mathcal{O}(G)/A)_0^! \in A_1$ is just the homology of $G = [\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}}$, which is trivial (by contractibility of $\mathbb{A}^1$ in the SH -formalism). Thus, the map $* \rightarrow M_1^*$ is an injection. Moreover, by Theorem 12.3 (i), the zero section $* \rightarrow M_1^*$ is 0-prim; thus, it is a closed immersion.

³⁴The converse is also true.

Let $\tilde{X} \subset M_1^*$ be the open complement of the zero section. Note that M_1^* itself is 0-suave, by another application of Theorem 12.3 (i), using that zero section $* \rightarrow G$ is 0-prim (and all maps $* \rightarrow B^k G$ are 1-prim for $k \geq 1$, by ambidexterity). It follows that $\tilde{X}$ is 0-suave over X . Moreover, one computes

$$\mathcal{O}(\tilde{X})_1 = \mathcal{O}(G)_1 / \mathcal{O}(X)_1 = \mathrm{SH}(\mathbb{A}_{\mathbb{Z}}^1) / \mathrm{SH}(\mathbb{Z})$$

the quotient under the fully faithful functor $\mathrm{SH}(\mathbb{Z}) \rightarrow \mathrm{SH}(\mathbb{A}_{\mathbb{Z}}^1)$ given by pullback. The functor

$$\mathrm{SH}(\mathbb{Z}) \rightarrow \mathcal{O}(\tilde{X})_1 = \mathrm{SH}(\mathbb{A}_{\mathbb{Z}}^1) / \mathrm{SH}(\mathbb{Z})$$

is given by inclusion along the zero section of $\mathbb{A}_{\mathbb{Z}}^1$. It is clear that this functor is conservative, so $\tilde{X} \rightarrow X$ is a cover by Proposition 8.10.

More generally, we see that for any scheme S of finite type over $\mathbb{Z}$, one can evaluate $\mathcal{O}(S \times_X \tilde{X})_1$ as

$$\mathrm{SH}(\mathbb{A}_S^1) / \mathrm{SH}(S)$$

where the symmetric monoidal structure combines the usual symmetric monoidal structure in the S -direction with the convolution symmetric monoidal structure in the $\mathbb{A}^1$ -direction. This is in fact the 6-functor formalism of “exponential motives”, cf. [FJ25], [GPL22].

Over $\tilde{X}$, we get a universal section of M_1^* , and hence a universal map

$$[\mathbb{A}_{\mathbb{Z}}^1]_{\tilde{X}} := [\mathbb{A}_{\mathbb{Z}}^1]_{\mathrm{SH}} \times_X \tilde{X} \rightarrow (\mathrm{GL}_1)_1;$$

this is the “exponential local system on $\mathbb{A}^1$ ”,

$$\mathcal{L} \in \mathcal{O}([\mathbb{A}_{\mathbb{Z}}^1]_{\tilde{X}})_1.$$

Under the above description of $\mathcal{O}([\mathbb{A}_{\mathbb{Z}}^1]_{\tilde{X}})_1$, this corresponds to the structure sheaf of the diagonal $\mathbb{A}_{\mathbb{Z}}^1$ inside $\mathbb{A}_{\mathbb{A}_{\mathbb{Z}}}^1$.

As M_1^* is a module over the commutative ring object G , the universal section of M_1^* induces a map

$$f : G \times_X \tilde{X} \rightarrow M_1^* \times_X \tilde{X}$$

over $\tilde{X}$. We want to see that this is an isomorphism. As both sides are 1-affine over $\tilde{X}$, it suffices to see that it induces an isomorphism after applying $(\mathcal{O}(-)/\mathcal{O}(\tilde{X}))_1$. Both sides evaluate to the object corresponding to $[\mathbb{A}_{\mathbb{Z}}^1] \in K_{\mathrm{SH}} = \mathcal{O}(X)_2$, base-changed to $\tilde{X}$. Thus, we have to see that a certain endomorphism of the image of $[\mathbb{A}_{\mathbb{Z}}^1]$ in $\mathcal{O}(\tilde{X})_2$ is an isomorphism. This endomorphism is in fact the one given by the kernel $m^* \mathcal{L}$, where $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the multiplication. Its inverse is given by a similar kernel; to check that the composites are correct, one only needs that the !-cohomology of $\mathcal{L}$ on $\mathbb{A}^1$ is trivial. But the !-cohomology of the diagonal $\mathbb{A}_{\mathbb{Z}}^1 \subset \mathbb{A}_{\mathbb{A}_{\mathbb{Z}}}^1$ along the first projection is just the constant sheaf on $\mathbb{A}_{\mathbb{Z}}^1$, which thus lies in the image of the constant sheaves, and is therefore zero in the Verdier quotient. $\square$

13. Lecture XIII: The Gestalt of Berkovich motives

In this last lecture, we discuss what is known about the geometry of one of the key motivating examples (for me): The Gestalt associated to the six-functor formalism of Berkovich motives. This lecture is meant to serve as an indication of how this story fits into the larger picture I am trying to develop; it is at many points extremely sketchy. Details will appear elsewhere.

Let C be the category of finite-dimensional Berkovich spaces. More precisely, in the language of [Sch25a], consider the category C of (light) arc-stacks X that admit a locally closed immersion into $\mathbb{A}_{\mathbb{Z}}^n$ for some $n \geq 0$. The key difference between the usual setting of algebraic geometry and the present setting of Berkovich geometry is that X is determined by inequalities $|f_i| \leq C_i$ for some functions f_i .³⁵ By [Sch25a], there is a six-functor formalism

$$D_{\text{mot}} : \text{Corr}(C) \rightarrow 1\text{Pr}_{D(\mathbb{Z})}$$

of étale Berkovich motives, with $\mathbb{Z}$ -coefficients. (One could also work with $\mathbb{S}$ -coefficients, and develop a non-étale theory; this is not my concern here.)

By [Sch25a, Proposition 10.3] (plus a small argument to extend to non-analytic Berkovich spaces), if X is quasicompact and lives over $\mathbb{Z}[\zeta_n]$ for some $n > 2$, then

$$D_{\text{mot}}(X)$$

is rigid as a symmetric monoidal presentable stable $(\infty, 1)$ -category. The assumption of living over $\mathbb{Z}[\zeta_n]$ with $n > 2$ ensures that all residue fields have finite cohomological dimension as opposed to just virtual finite cohomological dimension; the only obstruction here comes from possible real points. We note that like sheaves on a general compact Hausdorff space, $D_{\text{mot}}(X)$ is not compactly generated. Geometrically, there is a map

$$\text{Gest}(D_{\text{mot}}(X)) \rightarrow \text{Gest}(D(|X|, \mathbb{Z}))$$

where $|X|$ denotes the underlying compact Hausdorff space of X , and the fibres at $x \in |X|$ are given by $\text{Gest}(D_{\text{mot}}(K(x)))$ where $K(x)$ is the complete Banach field that is the residue field of X at x . These categories associated to Banach fields are compactly generated.

By our discussion of six-functor formalisms, we should however not focus on the Gestalt of $D_{\text{mot}}(X) \in \text{CAlg}(1\text{Pr})$, but rather of the presentable $(\infty, 2)$ -category of kernels $K_{D_{\text{mot}}}$, leading to the Gestalt

$$[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}} \in \text{Gest}/\text{Gest}(D(\mathbb{Z})).$$

Already in [Sch25a], I conjectured the following result:

THEOREM 13.1 (Aoki). *For any $n > 2$, the Gestalt*

$$[\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \in \text{Gest}/\text{Gest}(D(\mathbb{Z}))$$

is 0-proper.

SKETCH. For any six-functor formalism, the corresponding Gestalt is 1-proper, so we only need to worry about the right adjoints at the lowest categorical level. For the map

$$f : [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \rightarrow \text{Gest}(D(\mathbb{Z})),$$

³⁵It is a nontrivial matter to endow X with a good structure sheaf; we sidestep this issue here by working directly with arc-stacks. This means that there are more maps $X \rightarrow Y$ than one might a priori expect — for example if X lives in characteristic p , one can take arbitrary p -power roots of functions.

this means that

$$f_* : D_{\text{mot}}(\mathbb{Z}[\zeta_n]) \rightarrow D(\mathbb{Z})$$

is colimit-preserving. This is part of the rigidity of $D_{\text{mot}}(\mathbb{Z}[\zeta_n])$.

For the diagonal

$$\Delta_f : [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \times [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}},$$

we note that this map arises as follows. Consider the natural transformation

$$D_{\text{mot}}(X) \otimes_{D(\mathbb{Z})} D_{\text{mot}}(Y) \rightarrow D_{\text{mot}}(X \times Y)$$

of lax symmetric monoidal transformations

$$\text{Corr}(C') \times \text{Corr}(C') \rightarrow 1\text{Pr}_{D(\mathbb{Z})},$$

where C' is the subcategory of quasicompact objects of $C/\mathbb{Z}[\zeta_n]$. Then Δ_f arises by passing to modules over those lax symmetric monoidal transformations, considered as objects of

$$\text{CAlg}(\text{Fun}(\text{Corr}(C') \times \text{Corr}(C'), 1\text{Pr}_{D(\mathbb{Z})})).$$

Thus, we need to see that

$$D_{\text{mot}}(X) \otimes_{D(\mathbb{Z})} D_{\text{mot}}(Y) \rightarrow D_{\text{mot}}(X \times Y)$$

admits right adjoints, which are moreover functorial in $(X, Y) \in \text{Corr}(C') \times \text{Corr}(C')$. Using rigidity of $D_{\text{mot}}(Y)$, we can identify

$$D_{\text{mot}}(X) \otimes_{D(\mathbb{Z})} D_{\text{mot}}(Y) \cong \text{Fun}_{D(\mathbb{Z})}^L(D_{\text{mot}}(Y), D_{\text{mot}}(X)),$$

and then the right adjoint

$$D_{\text{mot}}(X \times Y) \rightarrow D_{\text{mot}}(X) \otimes_{D(\mathbb{Z})} D_{\text{mot}}(Y) \cong \text{Fun}_{D(\mathbb{Z})}^L(D_{\text{mot}}(Y), D_{\text{mot}}(X))$$

is given by associating to any $K \in D_{\text{mot}}(X \times Y)$ the corresponding functor with kernel K . This is evidently colimit-preserving, and moreover functorial in $X \in \text{Corr}(C')$ (which is just a condition). Breaking symmetry the other way, it is also functorial in Y .

It remains to see that the second diagonal Δ_{Δ_f} is 0-proper. For this, we make use of Theorem 9.12 (1). The map

$$\Delta_{\Delta_f} : [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \times_{[\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}} \times [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}}} [\mathcal{M}_{\text{arc}}(\mathbb{Z}[\zeta_n])]_{\text{mot}}$$

is then, as a moduli problem, given as follows. The target classifies, over a Gestalt S over $\text{Gest}(D(\mathbb{Z}))$, certain normed Ringgestalten (R, N) over S together with an automorphism $\gamma : R \rightarrow R$ commuting with the norm. The second diagonal corresponds to the subspace where $\gamma = \text{id}$. Note that as R is 0-truncated over S , this is really a condition. We want to see that Δ_{Δ_f} is a closed immersion. In other words, we need to see that the maximal subspace $S' \subset S$ over which $\gamma = \text{id}$ is closed. Let

$$Z = \{\gamma = \text{id}\} \subset R$$

be the fixpoint locus of γ inside R . This is closed, as it is the preimage of the closed immersion $\{0\} \subset R$ under the map $R \rightarrow R : x \mapsto \gamma(x) - x$. Thus, its complement $U \subset R$ is open (noting that we work over $\text{Gest}(D(\mathbb{S}))$, so open immersions and closed immersions are each other's complements). As R is 0-suave, it follows that also U is 0-suave, and hence its image $V \subset S$ is a 0-suave injection, i.e. an open immersion. Its complement $S' \subset S$ is therefore closed, and this is precisely the maximal sublocus where $Z \times_S S' = R \times_S S'$, i.e. where $\gamma = \text{id}$ as an automorphism of R . $\square$

We note that the proof combines the knowledge about D_{mot} from its explicit definition as in [Sch25a] to get the right adjoint for the first diagonal, with the moduli interpretation of $[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$ from [Aok25b] to get the right adjoint for the second diagonal. Working purely in terms of category theory, the second diagonal seems essentially inaccessible — the final argument really uses critically the geometrical perspective offered by Gestalten.

As a general intuition, it is reasonable to think of 0-proper Gestalten as being “compact Hausdorff”. Namely, being 0-prim implies being quasicompact by the next proposition, and being 0-proper then also includes similar quasicompactness conditions for the diagonals.

PROPOSITION 13.2. *Let X be a 0-prim Gestalt. Then X is quasicompact in the topos of n -étale Gestalten, for any $n \geq 0$. More precisely, if for some n there are n -suave maps $f_i : Y_i \rightarrow X$ for $i \in I$, such that $\bigsqcup_i Y_i \rightarrow X$ is surjective, then there is some finite subset $J \subset I$ for which $\bigsqcup_{i \in J} Y_i \rightarrow X$ is surjective.*

We prove this for $n = 1$, and indicate how to adapt the argument to higher values of n ; this requires the theory of n -retracts for all n .

PROOF. Replacing the Y_i by their images, we may assume that all f_i are n -étale injections, and then correspond to coidempotent coalgebras $C_i \in \mathcal{O}(X)_{n+1}$. For any finite $J \subset I$, we also get a coidempotent coalgebra C_J as the image of $\bigsqcup_{i \in J} Y_i \rightarrow X$. Then $\text{colim}_{J \subset I} C_J = 1$, so it suffices to see that if some filtered colimit of coidempotent coalgebras in $\mathcal{O}(X)_{n+1}$ is the unit, then already some term is the unit.

Thus, changing notation, take any filtered colimit

$$\text{colim}_i C_i = 1$$

of coidempotent coalgebras in $\mathcal{O}(X)_{n+1}$. Denote by $Y_i \subset X$ the Gestalt corresponding to C_i . If $n = 0$, then compactness of $1 \in \mathcal{O}(X)_1$ implies that 1 is a retract of C_i for some i , and then $C_i = 1$ as C_i is a coidempotent coalgebra. For $n > 0$, applying $\text{Hom}_{\mathcal{O}(X)_{n+1}}(1, -) : \mathcal{O}(X)_{n+1} \rightarrow n\text{Pr}$ and using n -primness, this implies that

$$\text{colim}_i \text{Hom}_{\mathcal{O}(X)_{n+1}}(1, C_i) \rightarrow \mathcal{O}(X)_n$$

is an isomorphism in $n\text{Pr}$. In particular, $1 \in \mathcal{O}(X)_n$ can be written as

$$\text{colim}_i f_{i,n-1,!}(D_i) \cong 1 \in \mathcal{O}(X)_n$$

for certain objects $D_i \in \text{Hom}_{\mathcal{O}(X)_{n+1}}(1, C_i)$. If $n = 1$, then 1 is a retract of $f_{i,n-1,!}(D_i)$ for some i , and then it follows from Proposition 8.11 (i) that $Y_i \rightarrow X$ is surjective, i.e. $Y_i = X$.

In general, one can iterate this argument, and the outcome is that for some i the map $C_i \rightarrow 1$ in $\mathcal{O}(X)_{n+1}$ is an $n+1$ -retract, i.e. there is some map $1 \rightarrow C_i$ (namely, the map given by D_i) such that the composite $1 \rightarrow 1$ is an n -retract. There is a version of Lemma 8.13 for $n+1$ -retracts, which shows that then 1 lies in the subcategory of $\mathcal{O}(X)_{n+1}$ of comodules over C_i , and hence $C_i = 1$. $\square$

Recall that there are basically two approaches to the theory of motives: An explicit one, via constructing many different realizations and studying their comparisons, and an abstract one, by building some universal category. We built $[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$ from the abstract point of view; connecting it to the concrete one means that we are looking for realizations, which in this context are maps

$$\text{Gest}(A) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$$

for “concrete” rings A . Here, concrete should ideally mean 0-affine, but we might also allow analytic rings, which are really just 1-affine. Let me pose the following conjecture.

CONJECTURE 13.3. *There is a cover*

$$X \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$$

where X is 1-affine and 0-proper over $\text{Gest}(D(\mathbb{Z}))$.

In other words, X corresponds to some rigid $\mathcal{O}(X)_1 \in \text{CAlg}(1\text{Pr}_{D(\mathbb{Z})})$. Rigid symmetric monoidal categories are usually relatively understandable, so a cover as the above would be a (somewhat) concrete realization of Berkovich motives.

Note also that the diagonal of $[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$ is 1-affine and 0-proper, so given any such X , we can recover $[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$ as the geometric realization of the Čech nerve $X_\bullet$, where all X_n are 1-affine and 0-proper. This Čech nerve is then taking care of “all comparison isomorphisms”. This would, at least qualitatively, connect the abstract theory of Berkovich motives to “concrete” realizations, and their comparison isomorphisms.

The following realizations of Berkovich motives are known:

- (i) The Betti realization. This is actually giving an isomorphism between the archimedean locus

$$[\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}} \times_{\mathcal{M}(\mathbb{Z})} \{|2| > 1\}$$

and $\text{Gest}(D((0, 1]/\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{Z}))$. Here, $(0, 1] \subset \mathcal{M}(\mathbb{Z})$ is the archimedean locus, with trivial action of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Namely, on the archimedean locus, the theory of Berkovich motives reduces to the usual theory of Betti sheaves, which already satisfies the Künneth formula, so the corresponding Gestalt is 1-affine.

- (ii) The analytic de Rham realization, cf. [ABL⁺25]. For any Banach field K of characteristic 0 (such as $\mathbb{Q}((t))$ or $\mathbb{Q}_p$), this gives a map

$$\text{AnSpec}(K) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$$

classifying the ring stack $(\mathbb{A}_K^1)_{\text{dR}}^{\text{an}}$. If $K = \mathbb{C}$, this is isomorphic to the Betti stack, as it must by the claimed universality in (i).

- (iii) The Hyodo–Kato realization, cf. [ABL⁺25]. For any perfectoid field C of characteristic 0 with an isomorphism $C^\flat \cong \mathbb{C}_p^\flat$, this gives a map

$$\text{AnSpec}(C) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$$

classifying the ring stack

$$[(\mathbb{A}_{\mathbb{C}_p}^1)^\flat_C]_{\text{dR}}^{\text{an}}.$$

Varying C , this gives, in fact, a map from the Fargues–Fontaine curve, or even the analytic de Rham stack of the Fargues–Fontaine curve,

$$(\text{FF}_{\mathbb{C}_p})_{\text{dR}}^{\text{an}}/\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}.$$

- (iv) The Habiro realization. Recall that the Habiro ring

$$\mathcal{H} = \lim_n \mathbb{Z}[q^{\pm 1}]/(q; q)_n, \quad (q; q)_n = \prod_{i=1}^n (1 - q^i),$$

is the completion of Laurent polynomials in q at all roots of unity. The analytic Habiro ring is a normed analytic ring, over the base solid normed ring $\mathbb{Z}((u))_{\square}$, given roughly as the initial $\mathbb{Z}((u))[q^{\pm 1}]_{\square}$ -algebra $\mathcal{H}^{\text{an}}$ such that

- (a) For all $k > 0$, the sequence $((q; q)_n / u^{kn})_n$ is a nullsequence.
- (b) For all $\epsilon > 0$, the sequence $(u^{\epsilon n} / (1 - q^n))_n$ is a nullsequence.

In particular, in $\mathcal{H}^{\text{an}}$, all $1 - q^n$ are inverted. Then I constructed an explicit ring stack over $\text{AnSpec}(\mathcal{H}^{\text{an}})$, yielding a map

$$\text{AnSpec}(\mathcal{H}^{\text{an}}) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}.$$

An important point here is that $\mathcal{H}^{\text{an}}/p \neq 0$; this gives a nontrivial map

$$\text{AnSpec}(\mathcal{H}^{\text{an}}/p) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Q})]_{\text{mot}} \times_{\text{Gest}(D(\mathbb{Z}))} \text{Gest}(D(\mathbb{F}_p)).$$

Conspicuously missing from this list is étale cohomology: Namely, while this does give a realization at the 1-categorical level, of $D_{\text{mot}}(X)$, it does not give a realization at the 2-categorical level. Still, this last example shows that one can hit at least part of the locus where one sees étale cohomology. In fact, one hits all of the equal characteristic 0 part:

PROPOSITION 13.4. *The map*

$$f : \text{AnSpec}(\mathcal{H}^{\text{an}}/p) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Q})]_{\text{mot}} \times_{\text{Gest}(D(\mathbb{Z}))} \text{Gest}(D(\mathbb{F}_p))$$

is surjective.

PROOF. Let us for simplicity assume $p \neq 2$; in general argue by descent from $\mathbb{Q}(\sqrt{-1})$. Then the target is 0-proper and the source is 0-prim, so the map is 0-prim. Thus, by Corollary 8.16, we have to see that $f_{0*}1 \in D_{\text{mot}}(\mathbb{Q}, \mathbb{F}_p)$ is descendable. But

$$D_{\text{mot}}(\mathbb{Q}, \mathbb{F}_p) = D(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{F}_p)$$

is the derived category of Galois representations. By finite cohomological dimension, any nonzero object in there is descendable. $\square$

From the explicit description of the Habiro ring stack, it is already defined over $\text{Gest}(\text{Nuc}(\mathcal{H}^{\text{an}}))$, the subcategory of nuclear $\mathcal{H}^{\text{an}}$ -modules; and these form a rigid category. Thus, we see that we can indeed explicitly cover

$$[\mathcal{M}_{\text{arc}}(\mathbb{Q})]_{\text{mot}} \times_{\text{Gest}(D(\mathbb{Z}))} \text{Gest}(D(\mathbb{F}_p)) \subset [\mathcal{M}_{\text{arc}}(\mathbb{Z})]_{\text{mot}}$$

by a 1-affine and 0-proper Gestalt, giving some evidence for Conjecture 13.3.

REMARK 13.5. The ring $\mathcal{H}^{\text{an}}/p$ is closely related to $\mathbb{F}_p((q-1))$, and via the theory of tilting this is connected to the theory of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -representations, i.e. local Galois representations. The previous proposition says that if one keeps track of the full descent datum along the surjective map

$$\text{AnSpec}(\mathcal{H}^{\text{an}}/p) \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{Q})]_{\text{mot}} \times_{\text{Gest}(D(\mathbb{Z}))} \text{Gest}(D(\mathbb{F}_p)),$$

one must get some “linear-algebraic” description of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representations, i.e. global Galois representations! It would be extremely interesting to understand explicitly what is happening here.

As another piece of evidence towards Conjecture 13.3, one has the following result.

THEOREM 13.6. *Fix some prime p . There is some 1-affine and 0-proper Gestalt X over $\text{Gest}(D(\mathbb{Q}))$ with a surjective map*

$$X \rightarrow [\mathcal{M}_{\text{arc}}(\mathbb{F}_p)] \times_{\text{Gest}(D(\mathbb{Z}))} \text{Gest}(D(\mathbb{Q})).$$

This is complementary to the previous result: Here the geometry is in characteristic p , but the coefficients are in characteristic 0. Using the invariance of Berkovich motives under tilting, one can formally extend the theorem to the whole locus $|p| < 1$ in $\mathcal{M}_{\text{arc}}(\mathbb{Z})$. Thus, in fibres over $\mathcal{M}(\mathbb{Z})$, one has the following information about Conjecture 13.3:

- (i) In the archimedean locus, it is true.
- (ii) Over the equal characteristic 0 point, it is true after profinite completion in the coefficients.
- (iii) Over the p -adic locus $|p| < 1$, it is true after rationalization in the coefficients.

Let us end this lecture by giving a very rough sketch of the ideas behind the proof of Theorem 13.6; an elaboration will appear elsewhere. In fact, the ideas may be helpful in attacking Conjecture 13.3 in full.

The starting point is an idea I extracted from the paper of Lahoti–Manam [LM25], that it may be a possibility to get a well-behaved moduli space of multiplicative monoid stacks (as opposed to ring stacks). The analogue of a norm in this setting is a metric — a norm N defines a metric d by $d(x, y) = N(x - y)$, but if we cannot add/subtract, we should keep the information of the metric. As we already understand the archimedean part, we focus on ultrametrics. A critical equation turns out to be

$$d(x^n, y^n) = \prod_{i=0}^{n-1} d(x, \zeta_n^i y) :$$

this is an equation that must be true for the multiplicative monoids underlying a normed ring stack equipped with all roots of unity. In order to formulate it, we need the roots of unity; thus, we will from now (without loss of generality) work over $\mathcal{M}_{\text{arc}}(\mathbb{Z}^{\text{cycl}})$. The universal multiplicative monoid $M := [\mathbb{A}_{\mathbb{Z}^{\text{cycl}}}^1]_{\text{mot}}$ with its metric $d(x, y) = N(x - y)$ and choice of roots of unity $\mathbb{Q}/\mathbb{Z} \rightarrow M$ has the following properties:

- (i) There is a zero $0 \in M$, and $0 \rightarrow M$ is a closed immersion, with open complement given by the units $M^\times \subset M$. The metrized monoid M is complete with respect to d .
- (ii) The monoid M is cohomologically smooth.
- (iii) The projective line

$$\mathbb{P}^1(M) := M \sqcup_{M^\times} M$$

is proper. Moreover, for any $x \in M$ and any $r \in (0, \infty)$, any closed disk

$$D(x, \leq r) = \{y \in M \mid d(x, y) \leq r\} \subset M, \quad D(x, \geq r) = \{y \in \mathbb{P}^1(M) \mid d(x, y) \geq r\}$$

is contractible.

- (iv) The map $N(x) = d(x, 0)$ is multiplicative; more generally, for all $x, y, z \in M$ one has $N(x)d(y, z) = d(xy, xz)$.
- (v) For all $x, y \in M$ and $n \geq 1$, one has

$$d(x^n, y^n) = \prod_{i=0}^{n-1} d(x, \zeta_n^i y).$$

Moreover, for n not a power of a prime, one has $d(1, \zeta_n) = 1$.

- (vi) The monoid M is divisible, and on the locus $d(1, \zeta_p) < 1$, the number p is invertible (on the Gestalt).

One can then consider the Gestalt Mon parametrizing ultrametrized multiplicative monoids M in Gestalten, with a zero and a chosen map $\mathbb{Q}/\mathbb{Z} \rightarrow M$, satisfying these axioms. It turns out that like Berkovich motives, this can be “computed” explicitly. The analogue of algebraically closed Banach fields are complete ultrametrized divisible multiplicative monoids with a zero and a chosen map from $\mathbb{Q}/\mathbb{Z}$. For each such monoid, one gets a “derived category of monoid motives” which in fact turns out to be generated by Tate twists. The underlying reason is that in this monoid geometry, one can only define simple rational varieties, whose cohomology is mixed-Tate. In joint work with Aoki, we will use this to show that Mon is also 2-affine and 0-proper.

CONJECTURE 13.7. *The map*

$$[\mathcal{M}_{\text{arc}}(\mathbb{Z}^{\text{cycl}})_{\text{na}}]_{\text{mot}} \rightarrow \text{Mon}$$

is surjective. Here, the subscript $_{\text{na}}$ indicates that one takes the nonarchimedean part.

As it is a 0-prim map, this reduces to descendability at the lowest categorical level. Moreover, endomorphisms of the unit on the target are the monoid-motives version of $D_{\text{mot}}(\mathcal{M}_{\text{arc}}(\mathbb{Z}^{\text{cycl}}))$, and combines sheaves on a compact Hausdorff space (which happens to be $\mathcal{M}(\mathbb{Z})_{\text{na}}$) with derived categories of monoid motives on “points”, which as noted above, are generated by Tate motives. This turns this descendability into a concrete statement; but it remains to compute the maps between Tate motives, which is a nontrivial matter.

The conjecture would concretely say that to construct normed ring stacks, yielding realizations of Berkovich motives, it is enough to construct metrized monoid stacks — any example will admit a compatible addition, at least after taking a cover of the base Gestalt. As the example of the Habiro ring stack shows, it is in fact often much easier to understand the underlying multiplicative monoid.

However, for our present purposes, the conjecture is not required. Namely, the map

$$[\mathcal{M}_{\text{arc}}(\mathbb{Z}^{\text{cycl}})_{\text{na}}]_{\text{mot}} \rightarrow \text{Mon}$$

is 0-proper and also 1-affine. Thus, to cover the source by 0-proper and 1-affine Gestalten, it suffices to so cover the target. But passing to the locus $d(1, \zeta_p) = 0$ and working rationally, one can show that there are actually no maps between different Tate twists in the theory of monoid motives (by a version of the standard argument for usual motives over $\mathbb{F}_p$, that Frobenius is both multiplication by a power of p and the identity), which means that any nontrivial map towards this locus will be a cover. But the ring stack $((\mathbb{A}_{\mathbb{C}_p}^1)_{\mathbb{C}_p}^{\sharp})_{\text{dR}}^{\text{an}}$, or also the one from [Sch25b, Lecture XII], gives us a nontrivial map. Incidentally, this argument shows that on this locus, Conjecture 13.7 is also true.

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